BRILLIANT POINTS OF CURVES AND SURFACES*

BY

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INTRODUCTION.

The geometrical investigations of this paper may best be introduced by a concrete optical illustration. We select the following one.

When a circular saw, which has been polished with emery while rotating in a lathe, is illuminated by a strong light—as for instance an electric arc—an observer will see in the saw a well defined curve of light. Photographs of this curve are shown in Fig. 1. The surface of the saw, with the circular scratches it has received by the particles of emery during the process of polishing, may be regarded as a corrugated surface of revolution. The curve of light is the assemblage of images of the electric arc in the corrugations of this reflecting surface. Neighboring images are very near together and therefore the assemblage of images resembles a continuous curve of light. In optics the points of light which have just been called images are called *brilliant points*. These points may be defined geometrically as follows:

Definition a. A point P is said to be a brilliant point of a surface t with respect to two points P_1 and P_2,† when the segments of right lines PP_1 and PP_2 lie in one plane through the normal to t at P and the angle P_1PP_2 is bisected *internally* by this normal. See Fig. 2.

In order to determine the geometric nature of the curve of light, let us make the following assumptions:

1) The corrugated surface which approximates the surface of the saw, may itself be approximated by tori ‡ of small radius.

2) A brilliant point of a tubular surface ‡ (in particular of a torus) of small radius may be replaced by the point which it approaches as the radius approaches zero.

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† P_1 and P_2 correspond to the point source of light and the eye of the observer respectively.

‡ A tubular surface is the envelope of a sphere of constant radius whose center describes a curve. The radius of the sphere is called the *radius* of the tubular surface, and the space curve its *axis*. A torus is a tubular surface with a circular axis.

In order to state the relation which defines the limit point of the second assumption, let us make

Definition b. A point P is said to be a brilliant point of the curve c with respect to the two points P_1 and P_2, when the segments of right lines PP_1 and PP_2 make *supplementary* angles with one of the directions of the tangent to c at P. See Fig. 2.

It is not difficult to convince one's self that *a brilliant point of a tubular surface of small radius approaches a brilliant point of the axis as the radius approaches zero.* The problem under consideration is thus reduced to

PROBLEM A. *Find the locus of the brilliant points of a family of concentric circles with respect to the points P_1 and P_2, which do not necessarily lie in the common plane of the circles.*

In the course of a solution of this problem it is desirable to consider also a point which is defined by replacing the word "supplementary" by the word

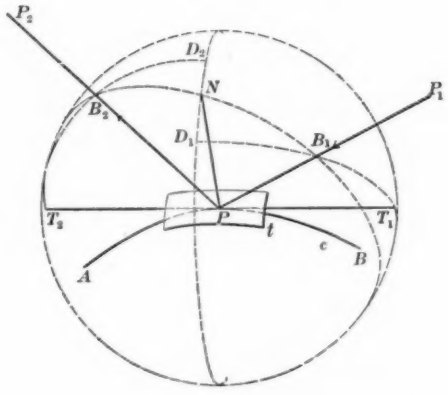


FIG. 2.

"equal" in Definition b. Such a point we will call a *virtual* brilliant point of the *curve* c with respect to P_1 and P_2.* Similarly, if we replace the word "internally" by the word "externally" in Definition a, we define what we will call a *virtual* brilliant point of the *surface* t with respect to P_1 and P_2. Let us call these two definitions the complements of Definitions b and a respectively.

In Fig. 3, which shows on the opposite sides of AB the two planes of descriptive geometry, the heavy full line represents the locus of brilliant points of

* When PP_1 and PP_2 are each perpendicular to the tangent to c at P, this definition and Definition b overlap. To be more precise, we apply the classification of § 1 to this case by putting $X_i = x - a_i$, $Y_i = y - b_i$, $Z_i = z - c_i$ $(i = 1, 2)$, where a_i, b_i, c_i are the coördinates of P_i $(i = 1, 2)$.

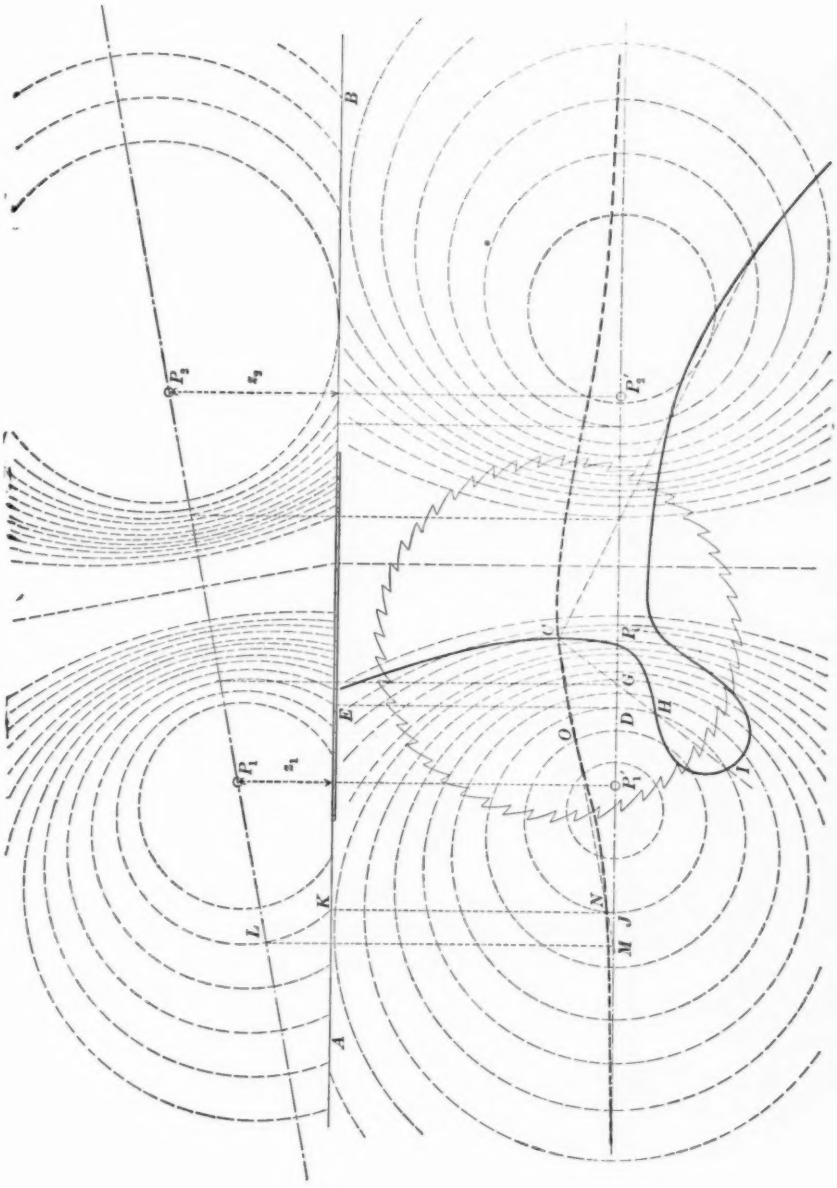


Fig. 3.

Problem A, and the heavy dashed line the locus of virtual brilliant points. The photographs in Fig. 1 do not show the locus of virtual brilliant points.*

While a curve or surface usually has only a finite number of brilliant points, there are exceptional cases. Every point of a $\left\{\begin{smallmatrix}\text{prolate spheroid}\\ \text{biparted hyperboloid of revolution}\end{smallmatrix}\right\}$ is a $\left\{\begin{smallmatrix}\text{brilliant point}\\ \text{virtual brilliant point}\end{smallmatrix}\right\}$ with respect to its foci. If an ellipse and hyperbola lie in perpendicular planes and are so placed that the vertices of each are the foci of the other, every point of either curve is a brilliant point or a virtual brilliant point with respect to any two points of the other.

Problem A may be generalized in several ways. One of the generalizations is the following:

PROBLEM B. *Find the locus of the brilliant points and virtual brilliant points of a two-parameter family of curves with respect to two points* P_1 *and* P_2.†

The term *congruence* is used to signify a two-parameter family of curves. All the straight lines which pass through the point P_1 (or P_2) form a rectilinear congruence. Problem B may be generalized by replacing the rectilinear congruences corresponding to P_1 and P_2 by two general curvilinear congruences.

In Part I (§§ 1–5) this generalized problem is stated, solved, and discussed. In Part II (§§ 6–8) the generalized problem of brilliant points of surfaces is discussed. In Part III (§§ 9, 10) the conditions are discussed under which some of the results of Parts I and II may be given special geometrical and optical interpretations.

I. BRILLIANT POINTS OF CURVES.

§ 1. *Definitions and classification.*

A congruence may be represented by the system of differential equations

$$\left\| \begin{matrix} dx & dy & dz \\ X & Y & Z \end{matrix} \right\| = 0, \ddagger \tag{1}$$

in which X, Y, Z: *a*) are single valued, continuous real functions of the rectangular Cartesian coördinates x, y, z; *b*) have first partial derivatives which are continuous; *c*) do not all vanish at the same point. Let Σ denote the region

* It is possible to find a family of curves such that if the surface of the saw were scratched along these instead of along the concentric circles formed by the emery particles during the process of polishing, the locus of the brilliant points would be the curve which for the circles is the locus of virtual brilliant points, and the locus of the virtual brilliant points would be the curve which for the circles is the locus of brilliant points. This is a consequence of the corollary to Theorem 7.

† For a solution of this problem, without derivation, together with some applications see a paper by the author, Annals of Mathematics, ser. 2, vol. 3 (1902), p. 113.

‡ This expresses that each of the three determinants of the matrix vanishes.

in which these conditions are satisfied. Through each point of Σ there passes one, and only one,* curve of the congruence (1).†

The two systems of differential equations

$$\begin{vmatrix} dx & dy & dz \\ X_i & Y_i & Z_i \end{vmatrix} = 0 \qquad (i=1,2), \tag{2}$$

in which X_i, Y_i, Z_i satisfy conditions $a), b), c)$ in Σ, represent two other congruences.

Definition 1. Let c be a curve with a continuously turning tangent which lies in Σ. A point P on c is said to be a brilliant point of c with respect to the two congruences (2), when the curves of the two congruences (2) which pass through P make equal angles with c at P.

In order to make a classification we will assign to each of the congruences (2) a *direction of propagation.* For this purpose we define $\overline{X}_i$, $\overline{Y}_i$, $\overline{Z}_i$ by the equations

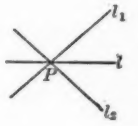
$$\frac{\overline{X}_i}{X_i} = \frac{\overline{Y}_i}{Y_i} = \frac{\overline{Z}_i}{Z_i} = \frac{1}{+\sqrt{X_i^2 + Y_i^2 + Z_i^2}} \qquad (i=1,2). \tag{3}$$

The directions of propagation for the two congruences (2) *shall be determined by the directional cosines* $\overline{X}_1$, $\overline{Y}_1$, $\overline{Z}_1$ *and* $-\overline{X}_2$, $-\overline{Y}_2$, $-\overline{Z}_2$ *respectively.*

Consider first the following special cases. In describing these, let l, l_1, l_2 represent respectively the tangents at P to the curve c and the curves of the two congruences (2) which pass through P.

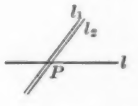
CASE I. The tangents l, l_1, l_2 lie in a plane:

1) l_1 and l_2 do not coincide;



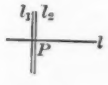
2) l_1 and l_2 coincide,

a) the coincident tangents are oblique to l,



b) the coincident tangents coincide with l,

c) the coincident tangents are perpendicular to l.

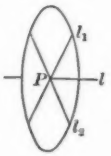


* The uniqueness follows from Condition b), which insures the LIPSCHITZ conditions.

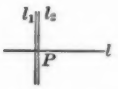
† For the purposes of this paper it is only necessary to know that a solution of (1) exists in the immediate neighborhood of a point of Σ.

CASE II. The tangents l_1 and l_2 are each perpendicular to the tangent l:

1) l_1 and l_2 do not coincide,



2) l_1 and l_2 coincide.*



A brilliant point which does not belong to either of these cases is said to belong to the general case. Brilliant points which belong to the general case or to the special case (I, 1) are called *actual* or *virtual* according as the directions of propagation at P of the two congruences (2) lie on the *same side* or on *opposite sides* of the normal plane to c at P. For convenience of reference let us make the following definitions:

Definition 2. Brilliant points which belong to the special case I are called *extra* brilliant points.

Definition 3. Brilliant points which belong to the special case II are called *exceptional* brilliant points.

Definition 4. Brilliant points which belong to the special cases I and II, with the exception of those which belong to the sub-case (I, 2, α), are called *extreme* brilliant points.

§ 2. *Conditions for brilliant points.*

Let us now find the condition that a point P of the region Σ should be a brilliant point of the curve of the congruence (1) which passes through it with respect to the two congruences (2).

The directions of propagation at P of the congruences (2) make with one of the directions of the curve of the congruence (1) which passes through P angles whose cosines are proportional to

$$X\overline{X}_i + Y\overline{Y}_i + Z\overline{Z}_i \qquad (i=1, 2).$$

Therefore the condition for a brilliant point is

$$X\overline{X}_1 + Y\overline{Y}_1 + Z\overline{Z}_1 = \pm (X\overline{X}_2 + Y\overline{Y}_2 + Z\overline{Z}_2), \tag{4}$$

with the sign $-$ for the actual, and the sign $+$ for the virtual brilliant point. This condition may be written as follows:

* The sub-cases (I, 2, c) and (II, 2) are identical.

$$(5)\qquad \begin{aligned}&[X(\bar{X}_1-\bar{X}_2)+Y(\bar{Y}_1-\bar{Y}_2)+Z(\bar{Z}_1-\bar{Z}_2)]\\&\quad\times[X(\bar{X}_1+\bar{X}_2)+Y(\bar{Y}_1+\bar{Y}_2)+Z(\bar{Z}_1+\bar{Z}_2)]=0.\end{aligned}$$

The two factors in (5) do not vanish together unless

$$(6)\qquad X:Y:Z=\begin{vmatrix}\bar{Y}_1-\bar{Y}_2 & \bar{Z}_1-\bar{Z}_2\\ \bar{Y}_1+\bar{Y}_2 & \bar{Z}_1+\bar{Z}_2\end{vmatrix}:\begin{vmatrix}\bar{Z}_1-\bar{Z}_2 & \bar{X}_1-\bar{X}_2\\ \bar{Z}_1+\bar{Z}_2 & \bar{X}_1+\bar{X}_2\end{vmatrix}:\begin{vmatrix}\bar{X}_1-\bar{X}_2 & \bar{Y}_1-\bar{Y}_2\\ \bar{X}_1+\bar{X}_2 & \bar{Y}_1+\bar{Y}_2\end{vmatrix},$$

but these equations are the conditions for the special cases after (I, 1).

Let the product of the left member of equation (5) and the non-vanishing expression

$$(X_1^2+Y_1^2+Z_1^2)\cdot(X_2^2+Y_2^2+Z_2^2)$$

be denoted by Ω. We shall find it convenient to express Ω in other forms. For this purpose let

$$U=\begin{vmatrix}Y_1 & Z_1\\ Y_2 & Z_2\end{vmatrix},\qquad V=\begin{vmatrix}Z_1 & X_1\\ Z_2 & X_2\end{vmatrix},\qquad W=\begin{vmatrix}X_1 & Y_1\\ X_2 & Y_2\end{vmatrix},$$

$$U_1=\begin{vmatrix}Y_2 & Z_2\\ Y & Z\end{vmatrix},\qquad V_1=\begin{vmatrix}Z_2 & X_2\\ Z & X\end{vmatrix},\qquad W_1=\begin{vmatrix}X_2 & Y_2\\ X & Y\end{vmatrix},$$

$$U_2=\begin{vmatrix}Y & Z\\ Y_1 & Z_1\end{vmatrix},\qquad V_2=\begin{vmatrix}Z & X\\ Z_1 & X_1\end{vmatrix},\qquad W_2=\begin{vmatrix}X & Y\\ X_1 & Y_1\end{vmatrix},$$

$$R=\begin{vmatrix}V & W\\ Y & Z\end{vmatrix},\qquad S=\begin{vmatrix}W & U\\ Z & X\end{vmatrix},\qquad T=\begin{vmatrix}U & V\\ X & Y\end{vmatrix},$$

$$t_i=XX_i+YY_i+ZZ_i \qquad (i=1,2).$$

Then

$$(7)\qquad \Omega=\begin{vmatrix}t_1^2 & X_1^2+Y_1^2+Z_1^2\\ t_2^2 & X_2^2+Y_2^2+Z_2^2\end{vmatrix}.$$

The right member of (7) may be written

$$\frac{\begin{vmatrix}t_1^2 & (R^2+S^2+T^2)(X_1^2+Y_1^2+Z_1^2)-(U^2+V^2+W^2)t_1^2\\ t_2^2 & (R^2+S^2+T^2)(X_2^2+Y_2^2+Z_2^2)-(U^2+V^2+W^2)t_2^2\end{vmatrix}}{R^2+S^2+T^2},$$

which, by means of the identities

$$(8)\qquad \begin{aligned}&(R^2+S^2+T^2)(X_i^2+Y_i^2+Z_i^2)-(U^2+V^2+W^2)t_i^2\\&\qquad=(RX_i+SY_i+TZ_i)^2 \quad (i=1,2),*\end{aligned}$$

*See footnote to formulas (25).

may be written

$$\frac{\begin{vmatrix} t_1 & RX_1+SY_1+TZ_1 \\ t_2 & RX_2+SY_2+TZ_2 \end{vmatrix} \cdot \begin{vmatrix} t_1 & -(RX_1+SY_1+TZ_1) \\ t_2 & RX_2+SY_2+TZ_2 \end{vmatrix}}{R^2+S^2+T^2}.$$

Hence, by the identity

$$R^2+S^2+T^2=\begin{vmatrix} t_1 & RX_1+SY_1+TZ_1 \\ t_2 & RX_2+SY_2+TZ_2 \end{vmatrix}, \tag{9}$$

*

$$\Omega = t_1(RX_2+SY_2+TZ_2)+t_2(RX_1+SY_1+TZ_1). \tag{10}$$

Since

$$\begin{cases} RX_1+SY_1+TZ_1=UU_2+VV_2+WW_2, \\ RX_2+SY_2+TZ_1=-[UU_1+VV_1+WW_1], \end{cases} \tag{11}$$

† †

we also have, from (10),

$$\Omega=\begin{vmatrix} UU_2+VV_2+WW_2 & XX_1+YY_1+ZZ_1 \\ UU_1+VV_1+WW_1 & XX_2+YY_2+ZZ_2 \end{vmatrix}. \tag{12}$$

‡

The proportion (6) may be written

$$X:Y:Z=U:V:W,$$

or

$$R=S=T=0. \tag{13}$$

The foregoing results may be expressed as follows:

THEOREM 1. *The necessary and sufficient condition that a point P of the region Σ should be a brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *is that $\Omega=0$ for P. Furthermore, when $R^2+S^2+T^2>0$, P is actual or virtual according as it causes the second or the first of the factors in equation* (5) *to vanish; and when $R=S=T=0$, P belongs to the special cases after* (I, 1).

* See footnote to formula (27).

† Different expansions of the determinants

$$\begin{vmatrix} U & V & W \\ X & Y & Z \\ X_i & Y_i & Z_i \end{vmatrix} \qquad (i=1,2).$$

‡ If, in the left member of (12), we put

$$X_i=x-a_i, \qquad Y_i=y-b_i, \qquad Z_i=z-c_i \qquad (i=1,2),$$

$$X=\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix}, \qquad Y=\begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial x} \end{vmatrix}, \qquad Z=\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix},$$

we obtain the left member of equation (b), p. 117, of the paper in the Annals.

§ 3. *Conditions for special cases.*

We will now take up a discussion of the special cases referred to at the end of § 1. This discussion will prove of fundamental importance when we come to the subject of brilliant points of surfaces.

Extra brilliant points. The necessary and sufficient condition that the three tangents l, l_1, l_2 should lie in a plane is

$$\Delta \equiv \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0. \tag{14}$$

Therefore the conditions for an extra brilliant point are

$$\Omega = 0, \qquad \Delta = 0. \tag{15}$$

These conditions may be expressed in another form by means of the identity

$$\begin{aligned} &\left[\begin{vmatrix} Y & Z \\ \overline{Y}_1+\overline{Y}_2 & \overline{Z}_1+\overline{Z}_2 \end{vmatrix}^2 + \begin{vmatrix} Z & X \\ \overline{Z}_1+\overline{Z}_2 & \overline{X}_1+\overline{X}_2 \end{vmatrix}^2 + \begin{vmatrix} X & Y \\ \overline{X}+\overline{X}_2 & \overline{Y}+\overline{Y}_2 \end{vmatrix}^2\right] \\ &\times\left[\begin{vmatrix} Y & Z \\ \overline{Y}_1-\overline{Y}_2 & \overline{Z}_1-\overline{Z}_2 \end{vmatrix}^2 + \begin{vmatrix} Z & X \\ \overline{Z}_1-\overline{Z}_2 & \overline{X}_1-\overline{X}_2 \end{vmatrix}^2 + \begin{vmatrix} X & Y \\ \overline{X}_1-\overline{X}_2 & \overline{Y}_1-\overline{Y}_2 \end{vmatrix}^2\right] \\ &= \frac{\Omega^2 + 4(X^2+Y^2+Z^2)(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)\Delta^2}{(X_1^2+Y_1^2+Z_1^2)^2\cdot(X_2^2+Y_2^2+Z_2^2)^2}, \end{aligned} \tag{16}$$

*

from which it follows that, when the equations (15) are satisfied, either

$$\begin{Vmatrix} X & Y & Z \\ \overline{X}_1+\overline{X}_2 & \overline{Y}_1+\overline{Y}_2 & \overline{Z}_1+\overline{Z}_2 \end{Vmatrix} = 0, \tag{17}$$

or

$$\begin{Vmatrix} X & Y & Z \\ \overline{X}_1-\overline{X}_2 & \overline{Y}_1-\overline{Y}_2 & \overline{Z}_1-\overline{Z}_2 \end{Vmatrix} = 0, \tag{18}$$

and conversely.

The system of equations (17) [or (18)] might be obtained as the solution of the pair of equations consisting of (14) and the equation obtained by equating to zero the first (or the second) factor of the left member of (5). In getting (17) and (18) in this way it is assumed that the two sets of equations

$$\overline{X}_1-\overline{X}_2=\overline{Y}_1-\overline{Y}_2=\overline{Z}_1-\overline{Z}_2=0, \quad \overline{X}_1+\overline{X}_2=\overline{Y}_1+\overline{Y}_2=\overline{Z}_1+\overline{Z}_2=0, \tag{19}$$

are *not* satisfied, i. e., that $U^2+V^2+W^2>0$.† But when the first (or second) of the sets of equations (19) is satisfied, equations (18) [or (17)] are satisfied.

* See footnote to formula (28).

† $[(\overline{X}_1-\overline{X}_2)^2+(\overline{Y}_1-\overline{Y}_2)^2+(\overline{Z}_1-\overline{Z}_2)^2]\cdot[(\overline{X}_1+\overline{X}_2)^2+(\overline{Y}_1+\overline{Y}_2)^2+(\overline{Z}_1+\overline{Z}_2)^2]$

$$=4\frac{U^2+V^2+W^2}{(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)}$$

identically. See footnote to formulas (21).

The foregoing results may be expressed as follows:

THEOREM 2. *The necessary and sufficient conditions that a point P of the region Σ should be an extra brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *may be expressed in either of the following two ways: a) the system of equations* (15) *is satisfied by P, b) either the system of equations* (17) *or the system of equations* (18) *is satisfied by P. Furthermore, when $U^2+V^2+W^2>0$, P belongs to case* (I, 1), *being an actual extra brilliant point when it satisfies* (18) *and a virtual extra brilliant point when it satisfies* (17); *and when $U=V=W=0$, P belongs to case* (I, 2).*

Exceptional Brilliant Points. The following theorem is evident:

THEOREM 3. *The necessary and sufficient conditions that a point P of the region Σ should be an exceptional brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *are that $t_i=0$, $(i=1,2)$ for P. Furthermore, when $U^2+V^2+W^2>0$, P belongs to case* (II, 1); *and when $U=V=W=0$, P belongs to case* (II, 2).

Extreme Brilliant Points. The following theorem will now be proved:

THEOREM 4. *The necessary and sufficient conditions that a point P of the region Σ should be an extreme brilliant point of the curve of the congruence* (1) *which passes through it, with respect to the two congruences* (2), *is that P satisfies the three equations*

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0.\dagger$$

Corollary. When the second of the congruences (2) coincides with the first throughout Σ, these equations become

$$t_1U_1=0, \qquad t_1V_1=0, \qquad t_1W_1=0.$$

Therefore either

$$t_1=0 \qquad \text{or} \qquad U_1=V_1=W_1=0.$$

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* For special cases (I, 2, *b*) and (I, 2, *c*) the conditions are

$$U=V=W=U_i=V=W_i=0 \qquad (i=1,2),$$

and

$$U=V=W=t_i=0 \qquad (i=1,2)$$

respectively. The last of these are given in Theorem 3, since the cases (I, 2, *c*) and (II, 2) are identical.

† Any one of these equations is dependent on the other two, since the left members satisfy the identity

$$X\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} + Y\begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} + Z\begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0$$

in virtue of the identities

$$XU_i+YV_i+ZW_i=0 \qquad =1,2).$$

The curves of the two congruences under consideration which pass through a point P, are perpendicular when P satisfies the equation $t=0$, and tangent when P satisfies the system of equations $U_1=V_1=W_1=0$.

In order to prove Theorem 4 it will be necessary to establish a few more identities.

From the well known identity

$$(20)\quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = (a_1^2+b_1^2+c_1^2)\cdot(a_2^2+b_2^2+c_2^2) - (a_1a_2+b_1b_2+c_1c_1)^2,$$

immediately follow the identities

$$(21)\quad \begin{cases} U_1^2+V_1^2+W_1^2=(X^2+Y^2+Z^2)(X_2^2+Y_2^2+Z_2^2)-t_2^2, \\ U_2^2+V_2^2+W_2^2=(X^2+Y^2+Z^2)(X_1^2+Y_1^2+Z_1^2)-t_1^2,* \end{cases}$$

$$(22)\quad R^2+S^2+T^2=(X^2+Y^2+Z^2)(U^2+V^2+W^2)-\Delta^2.$$

From (20) and the well known identities†

$$(23)\quad \begin{vmatrix} V_1 & W_1 \\ V_2 & W_2 \end{vmatrix}=X\Delta, \quad \begin{vmatrix} W_1 & U_1 \\ W_2 & U_2 \end{vmatrix}=Y\Delta, \quad \begin{vmatrix} U_1 & V_1 \\ U_2 & V_2 \end{vmatrix}=Z\Delta,$$

$$\begin{vmatrix} V_2 & W_2 \\ V & W \end{vmatrix}=X_1\Delta, \quad \begin{vmatrix} W_2 & U_2 \\ W & U \end{vmatrix}=Y_1\Delta, \quad \begin{vmatrix} U_2 & V_2 \\ U & V \end{vmatrix}=Z_1\Delta,$$

$$\begin{vmatrix} V & W \\ V_1 & W_1 \end{vmatrix}=X_2\Delta, \quad \begin{vmatrix} W & U \\ W_1 & U_1 \end{vmatrix}=Y_2\Delta, \quad \begin{vmatrix} U & V \\ U_1 & V_1 \end{vmatrix}=Z_2\Delta,$$

immediately follow the identities

$$(24)\quad [U_1U_2+V_1V_2+W_1W_2]^2=(U_1^2+V_1^2+W_1^2)(U_2^2+V_2^2+W_2^2) - (X^2+Y^2+Z^2)\Delta^2,$$

$$(25)\quad \begin{cases} [UU_1+VV_1+WW_1]^2=(U^2+V^2+W^2)(U_1^2+V_1^2+W_1^2) - (X_2^2+Y_2^2+Z_2^2)\Delta^2,‡ \\ [UU_2+VV_2+WW_2]^2=(U^2+V^2+W^2)(U_2^2+V_2^2+W_2^2) - (X_1^2+Y_1^2+Z_1^2)\Delta^2.‡ \end{cases}$$

*Similarly $U^2+V^2+W^2=(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)-(X_1X_2+Y_1Y_2+Z_1Z_2)^2$, which enables one to get the identity in the footnote to equations (19).

† From the theory of determinants.

‡ By (11) the left members become respectively

$$(RX_i+SY_i+TZ_i)^2 \qquad (i=2,1),$$

and by (21) and (22) the right members become respectively

$$(R^2+S^2+T^2)(X_i^2+Y_i^2+Z_i^2)-t_i^2(U^2+V^2+W^2) \qquad i=2,1),$$

whence follows (8).

The identities

$$\begin{aligned} t_1U_1 + t_2U_2 &= \Delta X - (X^2+Y^2+Z^2)U, \\ t_1V_1 + t_2V_2 &= \Delta Y - (X^2+Y^2+Z^2)V, \\ t_1W_1 + t_2W_2 &= \Delta Z - (X^2+Y^2+Z^2)W, \end{aligned} \tag{26}$$

are easily established. From them and (22) is obtained the identity

$$\begin{aligned} (t_1U_1 + t_2U_2)^2 + (t_1V_1 + t_2V_2)^2 + (t_1W_1 + t_2W_2)^2 \\ = (X^2+Y^2+Z^2)(R^2+S^2+T^2).* \end{aligned} \tag{27}$$

The product of the left member of (27) and the expression

$$(t_1U_1 - t_2U_2)^2 + (t_1V_1 - t_2V_2)^2 + (t_1W_1 - t_2W_2)^2$$

is

$$[t_1^2(U_1^2+V_1^2+W_1^2) + t_2^2(U_2^2+V_2^2+W_2^2)]^2 - 4t_1^2t_2^2[U_1U_2+V_1V_2+W_1W_2]^2$$

which by means of (24) becomes

$$[t_1^2(U_1^2+V_1^2+W_1^2) - t_2^2(U_2^2+V_2^2+W_2^2)]^2 + 4t_1^2t_2^2(X^2+Y^2+Z^2)\Delta^2.$$

By means of (21) and (7) the first term of this expression is $[(X^2+Y^2+Z^2)\Omega]^2$ and therefore we have the identity

$$\begin{aligned} [R^2+S^2+T^2]\cdot\left\{\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix}^2 + \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix}^2 + \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix}^2\right\} \\ = (X^2+Y^2+Z^2)\Omega^2 + 4t_1^2t_2^2\Delta^2\dagger. \end{aligned} \tag{28}$$

---

*Multiplying identities (26) by U, V, W and adding, we obtain

$$\begin{vmatrix} t_1 & -(UU_2+VV_2+WW_2) \\ t_2 & UU_1+VV_1+WW_1 \end{vmatrix} = \Delta^2 - (X^2+Y^2+Z^2)(U^2+V^2+W^2),$$

which by (11) and (22), becomes identity (9).

†In a similar manner identity (16) is obtained:

$$\left|\begin{matrix} Y \\ \bar{Y}_1 \pm \bar{Y}_2 \end{matrix} \quad \begin{matrix} Z \\ \bar{Z}_1 \pm \bar{Z}_2 \end{matrix}\right| = \frac{U_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}} \mp \frac{U_1}{\sqrt{X_2^2+Y_2^2+Z_2^2}},$$

$$\left|\begin{matrix} Z \\ \bar{Z}_1 \pm \bar{Z}_2 \end{matrix} \quad \begin{matrix} X \\ \bar{X}_1 \pm \bar{X}_2 \end{matrix}\right| = \frac{V_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}} \mp \frac{V_1}{\sqrt{X_2^2+Y_2^2+Z_2^2}},$$

$$\left|\begin{matrix} X \\ \bar{X}_1 \pm \bar{X}_2 \end{matrix} \quad \begin{matrix} Y \\ \bar{Y}_1 \pm \bar{Y}_2 \end{matrix}\right| = \frac{W_2}{\sqrt{X_1^2+Y_1^2+Z_1^2}} \mp \frac{W_1}{\sqrt{X_2^2+Y_2^2+Z_2^2}},$$

where the upper signs are read together and the lower are read together. The product of the

From identity (28) the following three lemmas are deduced:

Lemma I. If the point P of the region Σ satisfies the two equations

$$\Omega = 0, \qquad \Delta = 0, \tag{29}$$

and the inequality

$$U^2 + V^2 + W^2 > 0, \tag{30}$$

or if it satisfies the two equations

$$t_1 = 0, \qquad t_2 = 0, \tag{31}$$

it then also satisfies the three equations

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0. \tag{32}$$

Lemma II. If the point P of the region Σ satisfies the three equations (32) and the inequality (30), it then also satisfies either *a*) the two equations (29) and the inequality $t_1 t_2 \neq 0$, or *b*) the two equations (31) and the inequality $\Delta \neq 0$.

Lemma III. If the point P of the region Σ satisfies the three equations

$$U = V = W = 0, \tag{33}$$

it does *not* satisfy the three equations (32), *except* when either

$$U = V = W = U_i = V_i = W_i = 0 \qquad (i = 1, 2), \tag{34}$$

or

$$U = V = W = t_i = 0 \qquad (i = 1, 2). \tag{35}$$

The proofs follow:

When $\Delta = 0$ and $U^2 + V^2 + W^2 > 0$, it follows from (22) that $R^2 + S^2 + T^2 > 0$. Therefore it follows from (28) that, when equations (29) and inequality (30) are satisfied, equations (32) are satisfied. When $t_1 = t_2 = 0$, equations (32) are evidently satisfied. Hence Lemma I is proved.

---

sums of the squares for each reading of sign is

$$\left[\frac{U_2^2 + V_2^2 + W_2^2}{X_1^2 + Y_1^2 + Z_1^2} + \frac{U_1^2 + V_1^2 + W_1^2}{X_2^2 + Y_2^2 + Z_2^2}\right]^2 - 4\,\frac{[U_2U_1 + V_2V_1 + W_2W_1]^2}{(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)},$$

which by means of (24) becomes

$$\left[\frac{U_2^2 + V_2^2 + W_2^2}{X_1^2 + Y_1^2 + Z_1^2} - \frac{U_1^2 + V_1^2 + W_1^2}{X_2^2 + Y_2^2 + Z_2^2}\right]^2 + 4\,\frac{(X^2 + Y^2 + Z^2)\Delta^2}{(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)}.$$

By means of (21) and (7) the first term of this expression is

$$\frac{\Omega^2}{[(X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2)]^2}$$

and therefore we have identity (16).

In order to prove Lemma II it is first necessary to show that when equations (32) are satisfied, one of the two expressions t_1, t_2 cannot vanish unless the other vanishes also. For let us suppose either

$$t_1 = 0 \qquad \text{or} \qquad t_2 = 0.$$

Then equations (32) become either

$$t_2 U_2 = t_2 V_2 = t_2 W_2 = 0 \qquad \text{or} \qquad t_1 U_1 = t_1 V_1 = t_1 W_1 = 0$$

respectively. But by (21)

$$U_2^2 + V_2^2 + W_2^2 > 0 \qquad \text{or} \qquad U_1^2 + V_1^2 + W_1^2 > 0.$$

Therefore

$$t_2 = 0 \qquad \text{or} \qquad t_1 = 0.$$

Now let us suppose equations (32) to be satisfied. Then by (28) the equations

$$\Omega = 0 \qquad \text{and} \qquad t_1 t_2 \Delta = 0$$

are satisfied. Therefore, by the first part of this proof, either

$$\Omega = 0 \qquad \text{and} \qquad \Delta = 0$$

or

$$t_1 = 0 \qquad \text{and} \qquad t_2 = 0.*$$

In order to show that the system of equations $t_i = 0$ $(i = 1, 2)$, and the equation $\Delta = 0$ cannot be simultaneously satisfied when $U^2 + V^2 + W^2 > 0$, let us solve the system of equations

$$t_i = X_i X + Y_i Y + Z_i Z = 0 \qquad (i = 1, 2).$$

The solution is

$$X : Y : Z = U : V : W.$$

When $\Delta = 0$ and $U^2 + V^2 + W^2 > 0$, it follows from (22) that $R^2 + S^2 + T^2 > 0$. But from the solution, $R = S = T = 0$. Hence a contradiction, and therefore $t_1 \neq 0$, $t_2 \neq 0$.† When $t_i = 0$ $(i = 1, 2)$ and $U^2 + V^2 + W^2 > 0$, it follows that $R = S = T = 0$, whence by (22), $\Delta \neq 0$. Hence Lemma II is proved.

When $U = V = W = 0$, $\Delta = 0$ also, and therefore by (26),

$$t_1 U_1 + t_2 U_2 = t_1 V_1 + t_2 V_2 = t_1 W_1 + t_2 W_2 = 0.$$

Then the left members of equations (32) become either $2t_1 U_1$, $2t_1 V_1$, $2t_1 W_1$ or $-2t_2 U_2$, $-2t_2 V_2$, $-2t_2 W_2$. By (21), t_1 and $U_2^2 + V_2^2 + W_2^2$ cannot vanish together, and t_2 and $U_1^2 + V_1^2 + W_1^2$ cannot vanish together. Hence under the hypothesis $U = V = W = 0$, the equations (32) are not satisfied unless either $t_i = 0$ $(i = 1, 2)$ or $U_i = V_i = W_i = 0$ $(i = 1, 2)$.

From Lemmas I, II, III follows Theorem 4.

* Evidently $\Omega = 0$ when $t_1 = 0$ and $t_2 = 0$.

† By first part of proof, either t_1 and t_2 both vanish or neither vanishes.

§ 4. *Invariance of the condition for a brilliant point.*

It is evident that the expression Ω is invariant for any point transformation which preserves angles. An example of such a transformation is inversion with respect to a sphere.

It is not in this sense however that the term *invariance* is used in this article. It will be found that if each of the three congruences (1), (2) be replaced by certain other congruences (each involving an arbitrary function),

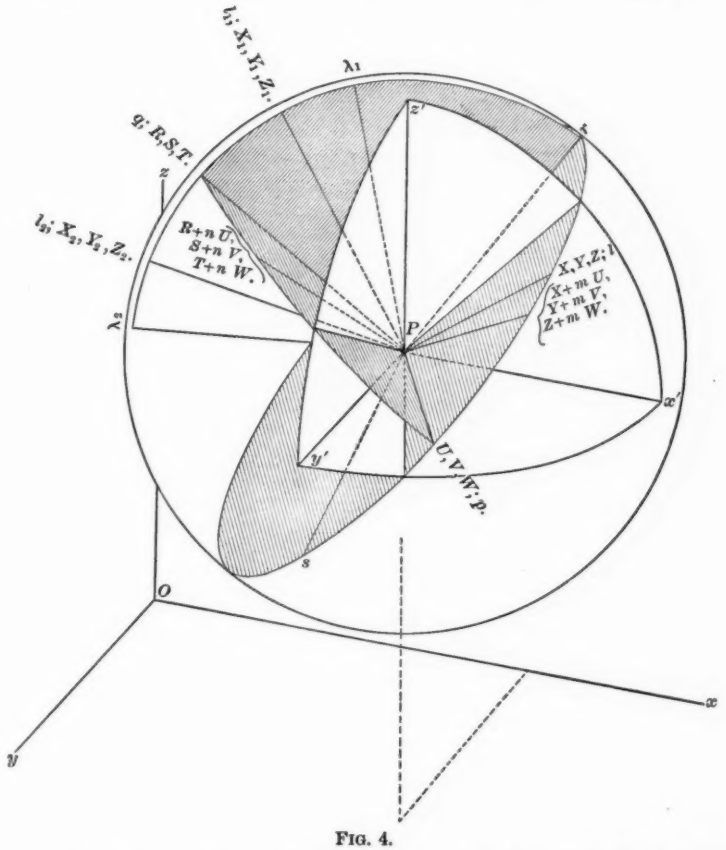


FIG. 4.

the expression Ω formed from the new congruences differs from that formed from the old ones by only a multiplicative factor.

Change of congruence (1) alone. In Fig. 4, P represents a point of the region Σ, and the right lines l and l_i $(i = 1, 2)$ represent tangents at P to the curves of the congruences (1) and (2) which pass through P. If l_1 and l_2 are distinct,

i. e., if $U^2+V^2+W^2>0$, the directional cosines of the bisectors of the angles formed by l_1 and l_2 are proportional to

$$\overline{X}_1-\overline{X}_2,\ \overline{Y}_1-\overline{Y}_2,\ \overline{Z}_1-\overline{Z}_2, \tag{36}$$

and

$$\overline{X}_1+\overline{X}_2,\ \overline{Y}_1+\overline{Y}_2,\ \overline{Z}_1+\overline{Z}_2. \tag{37}$$

Let π_1 and π_2 denote the planes which pass through P and are perpendicular respectively to these bisectors. Equation (5) is the condition that l should lie in one of the planes π_1, π_2. The perpendicular planes π_1 and π_2 intersect in the right line p, whose directional cosines are proportional to U, V, W. If l does not coincide with p, i. e., if $R^2+S^2+T^2>0$, the point P is an actual brilliant point or a virtual brilliant point according as l lies in π_2 or π_1. The expressions R, S, T are proportional to the directional cosines of the right line q which passes through P and is perpendicular to l and p. The three lines q, l_1, l_2 lie in one plane through P.

Let us now consider all the right lines which pass through P and lie either in the plane of l and p, or in the plane of p and q. The directional cosines of those which lie in the plane of l and p are proportional to

$$X+mU,\ Y+mV,\ Z+mW, \tag{38}$$

and of those which lie in the plane of p and q are proportional to

$$R+nU,\ S+nV,\ T+nW, \tag{39}$$

where m and n are parameters which may assume all real values. Each of these sets of lines excludes p. The planes of the two sets are perpendicular and intersect in p. The planes π_1 and π_2 are also perpendicular and intersect in p. Therefore, if l lies in $\pi_1(\pi_2)$ all the lines of the first set also lie in $\pi_1(\pi_2)$, and all those of the second set lie in $\pi_2(\pi_1)$; and if l lies in neither π_1 nor π_2 none of the lines of either set lie in π_1 or π_2. Let c denote the curve of congruence (1) which passes through P and therefore has l for a tangent at P, and γ and δ, curves which pass through P and have at P directional cosines proportional respectively to the expressions (38) and (39). Then follows from what has just been said: *If P is an actual (a virtual) brilliant point of c with respect to the two congruences* (2), *it is also an actual (a virtual) brilliant point of γ, and a virtual (an actual) brilliant point of δ, with respect to the two congruences* (2); *and if P is not a brilliant point of c it is not a brilliant point of γ or δ.*

In the portion of the region Σ for which $R^2+S^2+T^2>0$, let m and n in the expressions (38) and (39) now be regarded as functions of x, y, z which a) are single valued and continuous, b) have first partial derivatives which are con-

tinuous. Then each of the sets of functions (38), (39) satisfies in this region the conditions which the set of functions X, Y, Z satisfies in Σ. Therefore, for every choice of the functions m and n, each of the two systems of differential equations

$$\begin{vmatrix} dx & dy & dz \\ X+mU & Y+mV & Z+mW \end{vmatrix} = 0, \tag{40}$$

$$\begin{vmatrix} dx & dy & dz \\ R+nU & S+nV & T+nW \end{vmatrix} = 0, \tag{41}$$

represents a congruence. From the italicized statement above follows

THEOREM 5. *If, in the portion of the region Σ for which $R^2+S^2+T^2>0$, P $\{^{is}_{is\ not}\}$ a brilliant point of the curve of congruence* (1) *which passes through it with respect to the two congruences* (2), *then it* $\{^{is}_{is\ not}\}$ *a brilliant point of the curves of congruences* (40) *and* (41) *which pass through it with respect to the two congruences* (2), *for every choice of the functions m and n. Furthermore, if P is actual (virtual) for congruence* (1) *it is actual (virtual) for congruence* (40) *and virtual (actual) for congruence* (41).

Since the vanishing of Ω is the condition for a brilliant point of congruence (1) with respect to congruences (2) (see Theorem 1), Theorem 5 suggests that the substitution of the expressions (38) [or (39)] for X, Y, Z in Ω, results in an expression which differs from Ω by, at most, a factor which does not vanish when $R^2+S^2+T^2>0$. The expressions

$$XX_i+YY_i+ZZ_i \qquad (i=1,2)$$

are unaltered by the first substitution and changed into the expressions

$$RX_i+SY_i+TZ_i \qquad (i=1,2)$$

by the second substitution, since $UX_i+VY_i+WZ_i=0$ $(i=1,2)$. The expressions

$$RX_i+SY_i+TZ_i \qquad (i=1,2)$$

are unaltered by the first substitution and changed into the expressions

$$-(U^2+V^2+W^2)(XX_i+YY_i+ZZ_i) \qquad (i=1,2)$$

by the second substitution.* Therefore by (10) the expression Ω is unaltered

*Since

$$-\begin{vmatrix} R & U & X_i \\ S & V & Y_i \\ T & W & Z_i \end{vmatrix} = -(U^2+V^2+W^2)(XX+YY_i+ZZ_i).$$

See the next footnote.

by the first substitution and changed into the expression

$$-(U^2+V^2+W^2)\Omega \tag{42}$$

by the second substitution. From this analytic fact Theorem 5 might have been deduced.

The following are interesting corollaries of Theorem 5:

Corollary 1. When the function m is

$$\frac{-\Delta}{U^2+V^2+W^2},^*$$

every brilliant point of congruence (40) with respect to the congruences (2) is an extra brilliant point.

Corollary 2. When the function n is the constant zero, every brilliant point of the congruence (41) with respect to the congruences (2) is an extra brilliant point.

For, in the first case the new form of Δ is

$$\begin{vmatrix} X+mU & Y+mV & Z+mW \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \Delta + m(U^2+V^2+W^2),$$

which, for the given value of m, equals zero; and in the second case the new form if Δ is

$$\begin{vmatrix} R & S & T \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = RU+SV+TW = \begin{vmatrix} U & V & W \\ X & Y & Z \\ U & V & W \end{vmatrix} = 0.$$

Change of congruences (2) alone. Let us now consider the two right lines which pass through P (Fig. 4) and whose directional cosines are proportional to the expressions

$$f_1\overline{X}_1+f_2\overline{X}_2,\ f_1\overline{Y}_1+f_2\overline{Y}_2,\ f_1\overline{Z}_1+f_2\overline{Z}_2, \tag{43}$$

and

$$f_2\overline{X}_1+f_1\overline{X}_2,\ f_2\overline{Y}_1+f_1\overline{Y}_2,\ f_2\overline{Z}_1+f_1\overline{Z}_2, \tag{44}$$

in which f_1 and f_2 may have any real values not both zero. Let these lines be denoted by λ_1 and λ_2 respectively. They lie in the plane of l_1 and l_2. If l_1

* When m has this form, the expressions (38) are proportional to the directional cosines of the right line r which passes through P and is perpendicular to both p and q (see Fig. 4). For

$$\begin{vmatrix} S & V \\ T & W \end{vmatrix} = (U^2+V^2+W^2)X-\Delta U, \quad \begin{vmatrix} T & W \\ R & U \end{vmatrix} = (U^2+V^2+W^2)Y-\Delta V, \quad \begin{vmatrix} R & U \\ S & V \end{vmatrix} = (U^2+V^2+W^2)Z-\Delta W.$$

and l_2 are distinct and $f_1^2 - f_2^2 \neq 0$, λ_1 and λ_2 are distinct. In this case the directional cosines of the bisectors of the angles formed by λ_1 and λ_2 are proportional to the expressions

$$(45) \quad (f_1 - f_2)(\overline{X}_1 - \overline{X}_2),\ (f_1 - f_2)(\overline{Y}_1 - \overline{Y}_2),\ (f_1 - f_2)(\overline{Z}_1 - \overline{Z}_2),$$

and

$$(46) \quad (f_1 + f_2)(\overline{X}_1 + \overline{X}_2),\ (f_1 + f_2)(\overline{Y}_1 + \overline{Y}_2),\ (f_1 + f_2)(\overline{Z}_1 + \overline{Z}_2).$$

Comparing (45) and (46) with (36) and (37) respectively, we see that the bisectors of the angles formed by λ_1 and λ_2 are the same as those of the angles formed by l_1 and l_2. *Therefore the planes π_1 and π_2 are also perpendicular to the bisectors of the angles formed by λ_1 and λ_2.*

In the portion of the region Σ for which $U^2 + V^2 + W^2 > 0$, let f_1 and f_2 of the expressions (43) and (44) now be regarded as functions of x, y, z which *a*) are single valued and continuous, *b*) have first partial derivatives which are continuous, *c*) do not both vanish at the same point. Then each of the sets of functions (43), (44) satisfies in this region the conditions which the set of functions X, Y, Z satisfies in Σ. Therefore, for every choice of the functions f_1 and f_2, each of the two systems of differential equations

$$(47) \quad \begin{vmatrix} dx & dy & dz \\ f_1\overline{X}_1 + f_2\overline{X}_2 & f_1\overline{Y}_1 + f_2\overline{Y}_2 & f_1\overline{Z}_1 + f_2\overline{Z}_2 \end{vmatrix} = 0,$$

$$(48) \quad \begin{vmatrix} dx & dy & dz \\ f_2\overline{X}_1 + f_1\overline{X}_2 & f_2\overline{Y}_1 + f_1\overline{Y}_2 & f_2\overline{Z}_1 + f_1\overline{Z}_2 \end{vmatrix} = 0$$

represents a congruence.* From the last italicized statement follows

THEOREM 6. *If, in the portion of the region Σ for which $U^2 + V^2 + W^2 > 0$ and $f_1^2 - f_2^2 \neq 0$, $P\ \{^{is}_{is\ not}\}$ a brilliant point of the curve of congruence* (1) *which passes through it with respect to the two congruences* (2), *then it $\{^{is}_{is\ not}\}$ a brilliant point of the same curve with respect to the two congruences* (47) *and* (48), *for every choice of the functions f_1 and f_2.*

Theorem 6 may also be deduced in the following manner. If $U^2 + V^2 + W^2 > 0$, not all three of either of the sets of expressions (36), (37) vanish. The substitution of expressions (43) for $\overline{X}_1, \overline{Y}_1, \overline{Z}_1$, and expressions (44) for $\overline{X}_2, \overline{Y}_2, \overline{Z}_2$, changes the expressions (36) and (37) into the expressions (45) and (46) respectively, and therefore Ω into

$$(49) \quad (f_1^2 - f_2^2)\,\Omega.$$

If $f_1^2 - f_2^2 \neq 0$, (49) vanishes when, and only when, $\Omega = 0$. But by Theorem 1,

* If we regard congruences (2) as lines of force in fields of intensity f_1 and f_2 respectively, then congruence (47) gives the lines of force of the resultant field. By interchanging f_1 and f_2 we get congruence (48) as the lines of force of another field.

$\Omega = 0$ is condition that P should be a brilliant point of (1) with respect to (2). Hence Theorem 6.

Change of all three congruences. Let the directions of propagation of the two congruences (47) and (48) be determined by the senses of the vectors whose projections are the expressions (43) and (44) respectively.

Adopting this convention and uniting Theorems 5 and 6 we have

THEOREM 7. *If, in the portion of the region* Σ *for which* $R^2+S^2+T^2>0$ *and* $f_1^2-f_2^2 \neq 0$, P $\{^{is}_{is\ not}\}$ *a brilliant point of the curve of congruence* (1) *which passes through it with respect to the two congruences* (2), *then it* $\{^{is}_{is\ not}\}$ *a brilliant point of the curves of congruences* (40) *and* (41) *which pass through it with respect to the two congruences* (47) *and* (48), *for every choice of the four functions* m, n, f_1, f_2. *Furthermore, if* P *is actual* (*virtual*) *for congruence* (1) *with respect to the two congruences* (2), *it is actual* (*virtual*) *for congruence* (40) *and virtual* (*actual*) *for congruence* (41) *with respect to the two congruences* (47) *and* (48).

Let σ denote a surface composed of a one-parameter family of curves of congruence (1). We will call this family of curves the *first generation of* σ. Let P in Fig. 4 be a point of σ. The tangent plane to σ at P passes through l. This tangent plane cuts the plane of p and q in the right line λ. The right lines λ taken for all the points P of σ determine a second one-parameter family of curves on σ. We will call this family the *second generation of* σ. The second generation of σ is a one-parameter family of curves of congruence (41) for a particular function n. Therefore from Theorem 7 follows the

Corollary. In the portion of Σ for which $R^2+S^2+T^2>0$ and $f_1^2-f_2^2\neq 0$, the locus of the brilliant points of both the first and second generations of σ with respect to the two congruences (47) and (48), for every choice of the functions f_1 and f_2, is identical with that of the first generation of σ with respect to the two congruences (2). Furthermore, the portion of this locus which is composed of actual (virtual) brilliant points for the first generation of σ, is composed of virtual (actual) brilliant points for the second generation of σ.*

§ 5. *An illustrative example.*

For the three congruences (1), (2) let us take three families of radiating right lines. There will be no loss of generality in taking the "radiants" of these at the points $O=(0,0,0)$ and $P_i=(a_i,0,c)$ $(i=1,2)$. We shall assume

$$a_1-a_2\neq 0 \qquad \text{and} \qquad c\neq 0.$$

Then

$$X=x, \qquad Y=y, \qquad Z=z,$$

$$X_i=x-a_i, \qquad Y_i=y, \qquad Z_i=z-c, \qquad (i=1,2).$$

*The theorem stated on p. 121 of the paper in the Annals of Mathematics is a very special case of this corollary.

Thence follow

$$U=0, \qquad U_1=cy, \qquad U_2=-cy,$$

$$V=(a_1-a_2)(z-c), \quad V_1=a_2z-cx, \quad V_2=-a_1z+cx,$$

$$W=-(a_1-a_2)y, \qquad W_1=-a_2y, \qquad W_2=a_1y,$$

$$t_i=x^2+y^2+z^2-cz-a_ix \qquad (i=1,2),$$

$$R=(a_1-a_2)(y^2+z^2-cz), \quad S=(a_1-a_2)(-xy), \quad T=(a_1-a_2)(-xz+cx),$$

$$\Omega=(a_1-a_2)\cdot[-cx(z-c)\rho+(y^2+z^2-cz)\sigma],$$

$$\Delta=-(a_1-a_2)cy,$$

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix}=cy\rho, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix}=-cx\rho+z\sigma, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix}=-y\sigma,$$

in which

$$\rho=2(x^2+y^2+z^2-cz)-(a_1+a_2)x,$$

$$\sigma=(a_1+a_2)(x^2+y^2+z^2-cz)-2a_1a_2x,$$

and the determinant of the coefficients is $(a_1-a_2)^2\neq 0$. The region Σ includes all space except the three points O, P_1, P_2.

From the form of Ω it follows that the locus of brilliant points of the congruence of right lines of radiant O, with respect to the two points P_1, P_2, has the equation

$$(\alpha) \qquad \begin{aligned} &-cx(z-c)\cdot[2(x^2+y^2+z^2-cz)-(a_1+a_2)x] \\ &+(y^2+z^2-cz)\cdot[(a_1+a_2)(x^2+y^2+z^2-cz)-2a_1a_2x]=0.* \end{aligned}$$

The three points O, P_1, P_2, which are excluded from Σ, satisfy equation (α). The section of the surface (α) by the plane $y=0$ is the locus of extra brilliant points (since $\Delta=0$ when $y=0$). It consists of the right line

$$(\beta) \qquad y=0, \qquad z-c=0,$$

and the cubic

$$(\gamma) \qquad \begin{cases} y=0, \\ -2c(x^3+xz^2)+(a_1+a_2)(x^2z+z^3)+c(a_1+a_2)(x^2-z^2)+2(c^2-a_1a_2)xz=0. \end{cases}$$

In Fig. 5 the right line passing through P_1 and P_2 and the curve $AOQP_2OP_1B$ represent this locus. The curve (γ) is the locus of extra

* A method of constructing this surface is given below. A property of the surface is given in § 10.

brilliant points belonging to case (I, 1) (since for it $U^2+V^2+W^2>0$), and the line (β) is the locus of those belonging to case (I, 2) (since for it $U=V=W=0$). The section of the surface (α) by the plane $x=0$ is the locus of the exceptional brilliant points (since for it $t_1=t_2=0$). It is the circle

$$(\delta) \qquad x=0, \qquad y^2+z^2-cz=0.$$

This circle is a double line on the surface. It is the curve $OCQD$ in Fig. 5.

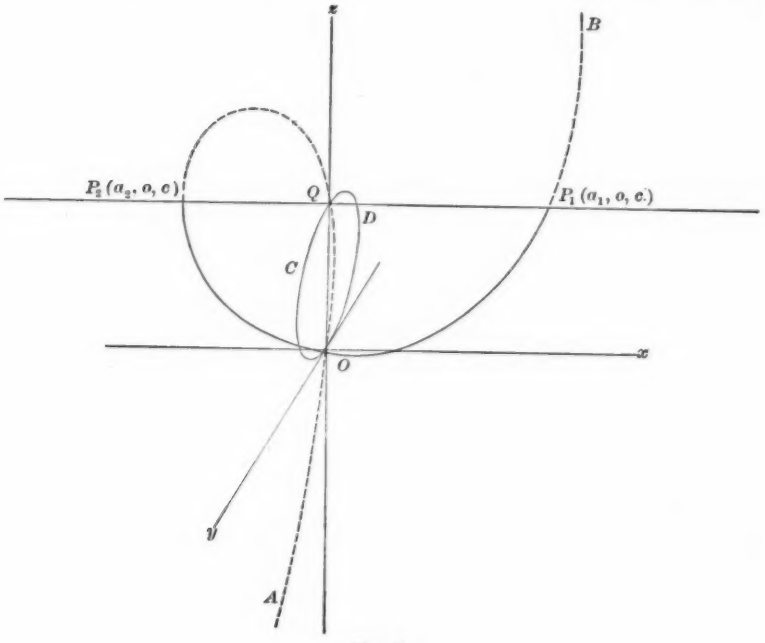


FIG. 5.

The locus of the extreme brilliant points is given by the three equations

$$cy\rho=0, \qquad -cx\rho+z\sigma=0, \qquad -y\sigma=0.$$

1) If $y\neq 0$, it follows from these that $\rho=0$ and $\sigma=0$, and therefore that

$$x=0, \qquad x^2+y^2+z^2-cz=0.$$

But these are the equations of the circle (δ).

2) If $y=0$, the equation $-cx\rho+z\sigma=0$ must be satisfied without ρ and σ necessarily being zero. But these equations are those of the cubic (γ). Therefore the locus of the extreme brilliant points consists of the cubic (γ) and the circle (δ).

For this example the systems of differential equations (40) and (41) are

$$(\epsilon) \qquad \left\| \begin{matrix} dx & dy & dz \\ x & y+(a_1-a_2)(z-c)\cdot m(x,y,z) & z-(a_1-a_2)y\cdot m(x,y,z) \end{matrix} \right\| = 0,$$

and

$$(\zeta) \qquad \left\| \begin{matrix} dx & dy & dz \\ y^2+z^2-cz & -xy+(z-c)\cdot n(x,y,z) & -x(z-c)-y\cdot n(x,y,z) \end{matrix} \right\| = 0,$$

respectively. $R^2 + S^2 + T^2 > 0$ for all space except the right line (β) and the circle (δ).

1) Let us suppose the function m to be of the form

$$\frac{cy}{(a_1 - a_2)\,[(z-c)^2 + y^2]}$$

(see Theorem 5, Cor. 1). Then (ϵ) becomes

$$(\eta) \qquad \left\| \begin{matrix} dx & dy & dz \\ x & y+\dfrac{cy(z-c)}{(z-c)^2+y^2} & z-\dfrac{cy^2}{(z-c)^2+y^2} \end{matrix} \right\| = 0.$$

The solution of (η) may be written in the form

$$(\theta) \qquad y = \kappa_1(z-c), \qquad \kappa_3 x = z - c\frac{\kappa_1^2}{1+\kappa_1^2},$$

in which κ_1 and κ_3 are the constants of integration. Equations (θ) represent the two-parameter family of right lines of which a one-parameter family lies in every plane through the right line (β), and all the right lines in any one of these planes pass through the point where this plane is pierced by the circle (δ).

2) Let the function n in (ζ) be identically zero. Then (ζ) becomes

$$(\iota) \qquad \left\| \begin{matrix} dx & dy & dz \\ y^2+z^2-cz & -xy & -x(z-c) \end{matrix} \right\| = 0.$$

The solution of (ι) may be written in the form

$$(\kappa) \qquad y = \kappa_1(z-c), \qquad x^2+y^2+z^2 = \kappa_2,$$

in which κ_1 and κ_2 are the constants of integration. Equations (κ) represent the two-parameter family of circles of which a one-parameter family lies in every plane through the right line (β), and all the circles in any one of these planes have as common center the point where this plane is pierced by the circle (δ).

3) Let the function m in (ϵ) be

$$\frac{x^2+y^2+z^2}{c(a_1-a_2)\cdot y} \qquad \left(\text{of the form } \frac{X^2+Y^2+Z^2}{-\Delta}\right).$$

Then (ϵ) becomes

$$(\lambda) \qquad \begin{vmatrix} dx & dy & dz \\ x & y + \dfrac{(x^2+y^2+z^2)(z-c)}{cy} & z - \dfrac{x^2+y^2+z^2}{c} \end{vmatrix} = 0.$$

The solution of (λ) may be written in the form

$$(\mu) \qquad x^2+y^2+z^2 = \kappa_2, \qquad \kappa_1 x + (x^2+y^2+z^2-cz) = 0,$$

in which κ_2 and κ_1 are the constants of integration. Equations (μ) represent the two-parameter family of circles which are the intersections of the spheres having the origin as center with the spheres which pass through the circle (δ).

By Theorem 5 the surface (α) is also the locus of the brilliant points of each of the two-parameter family of curves (θ), (κ), (μ) with respect to the two points P_1, P_2. By Corollaries 1 and 2 of Theorem 5, each of the brilliant points of the right lines (θ) and the circles (κ) is an extra brilliant point with respect to P_1 and P_2.

Construction of the surface (α). The locus of the brilliant points of a plane one-parameter family of radiating right lines with respect to two points P_1 and P_2, in the common plane of these lines, is easily constructed.* In Fig. 5 the cubic $AOQP_2OP_1B$ and the right line P_1P_2 is this locus for the right lines which pass through O and lie in the plane xOz, with respect to P_1 and P_2. This locus is the section of the surface (α) by the plane $y=0$. In order to get the section by any other plane through P_1P_2, as for instance the plane which passes through the point C of the circle $OCQD$, construct in this plane the locus of brilliant points of the right lines which pass through C, with respect to P_1 and P_2. This locus consists of the straight line P_1P_2 and a cubic curve of the same nature as $AOQP_2OP_1B$, with O replaced by C. This follows from the fact that the surface (α) is also the locus of the brilliant points of the two-parameter family of right lines (θ).

According to the convention in § 1, the direction of propagation of the congruence of right lines which pass through P_1, is *from* P_1, and that of those which pass through P_2, is *toward* P_2. Therefore the part of the curve $AOQP_2OP_1B$ (Fig. 5) which is drawn full is the locus of the actual brilliant points of the right lines which pass through O and lie in the plane xOz, and the dotted part is that of the virtual brilliant points. Similarly on the other sections of the surface (α) by the planes through P_1P_2, there are full and dotted parts which have the same significance.

*See page 128 of the paper in the Annals of Mathematics to which reference has already been made.

For this example the system of differential equations (47) and (48) are

$$(\nu)\qquad \begin{vmatrix} dx & dy & dz \\ f_1\dfrac{x-a_1}{r_1}+f_2\dfrac{x-a_2}{r_2} & f_1\dfrac{y}{r_1}+f_2\dfrac{y}{r_2} & f_1\dfrac{z-c}{r_1}+f_2\dfrac{z-c}{r_2} \end{vmatrix} = 0,$$

and

$$(\xi)\qquad \begin{vmatrix} dx & dy & dz \\ f_2\dfrac{x-a_1}{r_1}+f_1\dfrac{x-a_2}{r_2} & f_2\dfrac{y}{r_1}+f_1\dfrac{y}{r_2} & f_2\dfrac{z-c}{r_1}+f_1\dfrac{z-c}{r_2} \end{vmatrix} = 0,$$

respectively, where

$$r_i = +\sqrt{(x-a_i)^2+y^2+(z-c)^2} \qquad (i=1,2).$$

It is easy to see that the integral curves of both (ν) and (ξ) lie in the one-parameter family of planes

$$(o)\qquad y=\kappa_1(z-c).$$

By means of this relation, r_i becomes

$$+\sqrt{(x-a_i)^2+\left(1+\frac{1}{\kappa_1^2}\right)y^2} \qquad (i=1,2).$$

Then the differential equation for determining the integral curves of (ν) in any one of the planes (o) is

$$(\pi)\qquad \begin{vmatrix} dx & d\eta \\ f_1\dfrac{x-a_1}{\sqrt{(x-a_1)^2+\eta^2}}+f_2\dfrac{x-a_2}{\sqrt{(x-a_2)^2+\eta^2}} & f_1\dfrac{\eta}{\sqrt{(x-a_1)^2+\eta^2}}+f_2\dfrac{\eta}{\sqrt{(x-a_2)^2+\eta^2}} \end{vmatrix}=0,$$

where

$$\eta=\sqrt{1+\frac{1}{\kappa_1^2}}\,y$$

and represents the perpendicular distance from the right line P_1P_2. If now we put

$$(\rho)\qquad \cos\omega_i=\frac{x-a_i}{\sqrt{(x-a_i)^2+\eta^2}}, \qquad \sin\omega_i=\frac{\eta}{\sqrt{(x-a_i)^2+\eta^2}},$$

equation (π) becomes

$$(\sigma)\qquad \begin{vmatrix} d\eta & f_1\sin\omega_1+f_2\sin\omega_2 \\ dx & f_1\cos\omega_1+f_2\cos\omega_2 \end{vmatrix}=0.$$

From equations (ρ) we find

$$(\tau)\qquad x-\frac{a_1+a_2}{2}=\frac{a_1-a_2}{2}\cdot\frac{\sin(\omega_1+\omega_2)}{\sin(\omega_1-\omega_2)}, \qquad \eta=\frac{a_1-a_2}{2}\cdot\frac{2\sin\omega_1\sin\omega_2}{\sin(\omega_1-\omega_2)}.$$

If we make these substitutions in (σ), we get the differential equation

$$(\upsilon) \qquad \frac{a_1 - a_2}{\sin(\omega_1 - \omega_2)}\left[f_1 \sin\omega_2\, d\omega_1 + f_2 \sin\omega_1\, d\omega_2\right] = 0,$$

or

$$(\upsilon_1) \qquad f_1 r_1 d\omega_1 + f_2 r_2 d\omega_2 = 0,$$

since

$$r_1 = (a_1 - a_2)\frac{\sin\omega_2}{\sin(\omega_1 - \omega_2)}, \qquad r_2 = (a_1 - a_2)\frac{\sin\omega_1}{\sin(\omega_1 - \omega_2)}.$$

To get the corresponding equation for (ξ), interchange f_1 and f_2 in (υ_1).

Let us now find the solutions of (ν) and (ξ) when

$$(\phi) \qquad f_1 = \frac{M_1}{r_1} \qquad \text{and} \qquad f_2 = \frac{M_2}{r_2},$$

where M_1 and M_2 are constants. Then f_1 and f_2 are independent of κ_1. For these functions, the solution of equation (υ_1) is

$$(\chi) \qquad M_1\omega_1 + M_2\omega_2 = K,$$

where K is the constant of integration. If we interchange f_1 and f_2 in (υ), we get the equation

$$M_2 \csc^2\omega_1\, d\omega_1 + M_1 \csc^2\omega_2\, d\omega_2 = 0,$$

whence

$$(\psi) \qquad M_2 \operatorname{ctn}\omega_1 + M_1 \operatorname{ctn}\omega_2 = C,$$

where C is the constant of integration. From (ρ)

$$\operatorname{ctn}\omega_i = \frac{x - a_i}{\eta} \qquad (i = 1, 2),$$

and therefore (ψ) becomes

$$(\psi_1) \qquad C\eta = (M_1 + M_2)x - (M_1 a_2 + M_2 a_1).$$

Therefore for the functions (ϕ) the solution of (ν) is given by the equations (χ) and (o), and that of (ξ) by the equations (ψ) and (o). That is:

1) For the functions (ϕ), the integral curves of (ν) are the lines of force of a field due to two centers of force situated at P_1 and P_2, the intensity of the force due to each center being proportional to the reciprocal of the distance from that center.*

2) For the functions (ϕ), the integral curves of (ξ) are the right lines which pass through the point O' on P_1P_2, at which the force of the above field is zero.

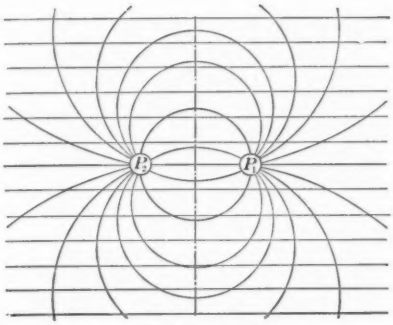
*For a method of making these curves physically evident see a note by the author: Bulletin of American Mathematical Society, vol. 12 (1906), p. 425.

For the functions (ϕ), $f_1^2 - f_2^2 \neq 0$ everywhere except on the sphere

$$(\omega) \qquad M_1^2 r_2^2 - M_2^2 r_1^2 = 0 .$$

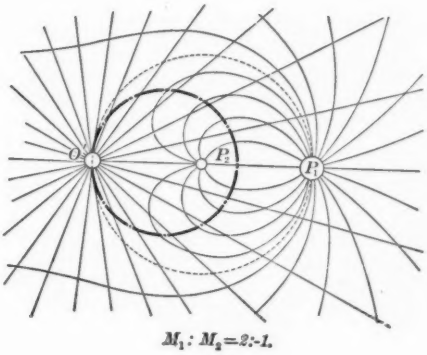
This sphere has P_1 and P_2 as a pair of conjugate points, and passes through the point O'.

Figs. 6, 7, 8, 9 show the integral curves of (ν) and (ξ) which lie in a plane



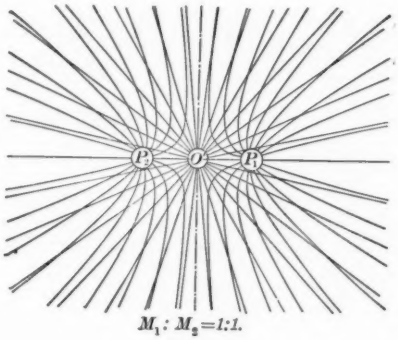
$M_1 : M_2 = 1 : -1.$

FIG. 6.



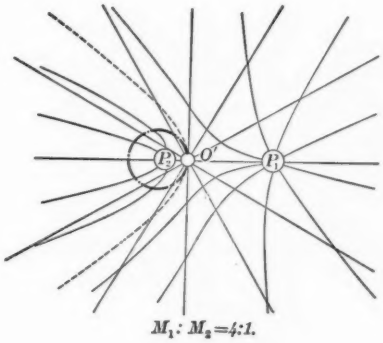
$M_1 : M_2 = 2 : -1.$

FIG. 8.



$M_1 : M_2 = 1 : 1.$

FIG. 7.



$M_1 : M_2 = 4 : 1.$

FIG. 9.

through P_1P_2 for the particular cases in which M_1 and M_2 of the particular functions (ϕ) have the values indicated. The dotted-and-dashed curve is a section of the sphere (ω) by a plane through P_1P_2.

By Theorem 7, the surface (α) is also the locus of the brilliant points of the right lines through O, and of each of the congruences (θ), (κ), and (μ), with respect to each of the pairs of congruences represented in section in Figs. 6, 7, 8, 9. The sphere shown in section by the dotted-and-dashed curve is excluded

from the region Σ. The direction of propagation of the lines of force being *from* P_1, and that of the right lines through O' being *toward* O', the distribution of the actual and virtual brilliant points is the same with respect to these congruences as with respect to P_1 and P_2.

II. Brilliant Points of Surfaces.

§ 6. *Definitions, classification and conditions.*

As a generalization of Definition a and its complement (see Introduction) we have the following:

Definition 5. Let t be a surface with a continuously turning tangent plane which lies in Σ. A point P on t is said to be a brilliant point of t with respect to the two congruences (2) when the directions of propagation of the two congruences (2) at P lie in one plane through the normal to t at P and the angle formed at P by these two directions is bisected either externally or internally by this normal.*

This definition does not include the cases in which the directions of propagation of the two congruences (2) at P are equal or opposite along any right line through P, except the normal and a tangent line.

If, in the figures of § 1, l_1 and l_2 are tangents at P to the curves of the two congruences (2) which pass through P and l is the normal to t at P, then all the possibilities of Definition 5 are shown in the first, third and fourth figures. Of the two following definitions, the first includes, besides the possibilities of Definition 5, those shown in the second figure; and the second includes, besides the possibilities of Definition 5, those shown in the fifth figure.

Definitions 6. Let t be a surface with a continuously turning tangent plane which lies in Σ. A point P on t is said to be a $\{\substack{first\\second}\}$ *extended* brilliant point of t with respect to the two congruences (2), when it is an $\{\substack{extra\\extreme}\}$ brilliant point of a curve c which cuts the surface t orthogonally at P, with respect to the two congruences (2).

The point P is said to be an actual or a virtual brilliant point of t according as it is a virtual or an actual brilliant point of c.

A one-parameter family of surfaces may be represented by the equation

$$F(x, y, z) = C, \tag{50}$$

in which C represents the parameter. Let us assume that the first partial derivatives of F, namely F_x, F_y, F_z, satisfy in Σ the same conditions which

* If we apply this definition to the case in which $X_i = x - a_i$, $Y_i = y - b_i$, $Z_i = z - c_i$, where a_i, b_i, c_i are the coördinates of P_i $(i = 1, 2)$, we get Definition a and its complement.

X, Y, Z satisfy. Then the system of differential equations

$$\begin{Vmatrix} dx & dy & dz \\ F_x & F_y & F_z \end{Vmatrix} = 0 \tag{51}$$

represents the congruence of curves which cut the surfaces (50) orthogonally.

From Definitions 6 and Theorems 2 and 4 the following theorems are immediately deduced:

THEOREM 8. *The necessary and sufficient conditions that a point P of the region Σ should be a first extended brilliant point of that one of the surfaces* (50) *which passes through P, with respect to the two congruences* (2), *may be expressed in either of the following two ways: a) the point P satisfies the two equations*

$$\underline{\Omega} = 0, \qquad \underline{\Delta} = 0, \tag{52}$$

where $\underline{\Omega}$ and $\underline{\Delta}$ are the expressions obtained from Ω and Δ respectively by replacing X, Y, Z by F_x, F_y, F_z respectively; b) the point P satisfies either the system of equations

$$\begin{Vmatrix} F_x & F_y & F_z \\ \overline{X}_1 + \overline{X}_2 & \overline{Y}_1 + \overline{Y}_2 & \overline{Z}_1 + \overline{Z}_2 \end{Vmatrix} = 0, \tag{53}$$

or the system of equations

$$\begin{Vmatrix} F_x & F_y & F_z \\ \overline{X}_1 - \overline{X}_2 & \overline{Y}_1 - \overline{Y}_2 & \overline{Z}_1 - \overline{Z}_2 \end{Vmatrix} = 0. \tag{54}$$

Furthermore, when $U^2 + V^2 + W^2 > 0$, P is an actual or a virtual brilliant according as it satisfies (53) *or* (54).

THEOREM 9. *The necessary and sufficient conditions that a point P of the region Σ should be a second extended brilliant point of that one of the surfaces* (50) *which passes through P, with respect to the two congruences* (2), *is that P satisfies the three equations*

$$\begin{vmatrix} t_1 & U_2 \\ t_2 & U_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & V_2 \\ t_2 & V_1 \end{vmatrix} = 0, \qquad \begin{vmatrix} t_1 & W_2 \\ t_2 & W_1 \end{vmatrix} = 0, \tag{55}$$

*where t_i, U_i, V_i, W_i are the expressions obtained from t_i, U_i, V_i, W_i $(i = 1, 2)$ respectively by replacing X, Y, Z by F_x, F_y, F_z respectively.**

We may now easily deduce

THEOREM 10. *The necessary and sufficient conditions that a point P of the region Σ should be a brilliant point of that one of the surfaces* (50) *which*

*See corollary and footnote to Theorem 4.

passes through P, with respect to the two congruences (2), *may be expressed in either of the following two ways:*

a) The point P satisfies either the equations (52) [*or* (53) *or* (54)] *and the inequality $U^2+V^2+W^2>0$, or one of the systems of equations*

$$U=V=W=\underline{U}_i=\underline{V}_i=\underline{W}_i=0 \quad (i=1,2), \qquad U=V=W=\underline{t}_i=0 \quad (i=1,2).$$

b) The point P satisfies either the equations (55) *and the inequality $\underline{t}_1\cdot\underline{t}_2\neq 0$, or the system of equations $U=V=W=\underline{t}_1=\underline{t}_2=0$.*

§ 7. *Invariance of the conditions for brilliant points.*

In § 4 it was shown that the expression Ω is changed into the expression (49) by the substitution of the expressions (43) for X_1, Y_1, Z_1 and the expressions (44) for X_2, Y_2, Z_2. The same substitution changes the expression Δ into the expression

$$\kappa(f_1^2-f_2^2)\Delta,$$

where κ is the non-vanishing expression $1/\sqrt{(X_1^2+Y_1^2+Z_1^2)(X_2^2+Y_2^2+Z_2^2)}$. As the analogon of Theorem 6 we therefore have

THEOREM 11. *If, in the portion of the region Σ for which $U^2+V^2+W^2>0$ and $f_1^2-f_2^2\neq 0$, P $\left\{\begin{smallmatrix}\text{is}\\ \text{is not}\end{smallmatrix}\right\}$ a brilliant point (second extended brilliant point) of that one of the surfaces* (50) *which passes through P, with respect to the two congruences* (2), *then it $\left\{\begin{smallmatrix}\text{is}\\ \text{is not}\end{smallmatrix}\right\}$ a brilliant point (second extended brilliant point) of the same surface with respect to the two congruences* (47) *and* (48), *for every choice of the functions f_1 and f_2.*

§ 8. *An example.*

For the one-parameter family of surfaces (50) let us take the concentric spheres

$$(\varpi) \qquad x^2+y^2+z^2=C,$$

of which the common center is the origin $O=(0,0,0)$. The orthogonal congruence (51) is the two-parameter family of right lines which pass through O. The loci of the extra and extreme brilliant points of this congruence, with respect to the two points P_1 and P_2, were found in § 5. The locus of the extra brilliant points consists of the right line (β) and the cubic (γ). By Theorem 8 these curves are also the locus of the first extended brilliant points of the spheres (ϖ) with respect to P_1 and P_2. The locus of the extreme brilliant points of the right lines through O consists of the cubic (γ) and the circle (δ). By Theorem 9 these curves are also the locus of the second extended brilliant points of the spheres (ϖ) with respect to P_1 and P_2. By Theorem 10 the cubic (γ) is

the locus of the brilliant points of the spheres (ϖ) with respect to P_1 and P_2. The portion of the cubic (γ) which is drawn dotted in Fig. 5 is the locus of the actual, and that which is drawn full, the locus of the virtual brilliant points of the spheres (ϖ).

By Theorem 11, in the portion of space which excludes the right line (β) and the locus $f_1^2 - f_2^2 = 0$, the curve (γ) and the pair of curves (γ), (δ) are the loci of the brilliant points and second extended brilliant points respectively of the spheres (ϖ), with respect to the two congruences (ν) and (ξ), for every choice of the functions f_1 and f_2. Four particular cases of the pairs of congruences (ν), (ξ) are represented in Figs. 6, 7, 8, 9, which have already been described.

III. Special Properties.

§ 9. *Brilliant points as points of contact.*

In the portion of the region Σ for which $U^2 + V^2 + W^2 > 0$, each of the three systems of differential equations

$$(56)\qquad \left\| \begin{matrix} dx & dy & dz \\ \overline{X}_1 - \overline{X}_2 & \overline{Y}_1 - \overline{Y}_2 & \overline{Z}_1 - \overline{Z}_2 \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} dx & dy & dz \\ \overline{X}_1 + \overline{X}_2 & \overline{Y}_1 + \overline{Y}_2 & \overline{Z}_1 + \overline{Z}_2 \end{matrix} \right\| = 0,$$

$$(57)\qquad \left\| \begin{matrix} dx & dy & dz \\ U & V & W \end{matrix} \right\| = 0,$$

represents a congruence.

Let us suppose that both of the congruences (56) are *normal congruences** and that the corresponding families of orthogonal surfaces are represented by the equations

$$(58)\qquad F_1(x, y, z) = C_1 \qquad \text{and} \qquad F_2(x, y, z) = C_2$$

respectively. The planes π_1 and π_2 defined in § 4 are the tangent planes at $P = (\xi, \eta, \zeta)$ to the surfaces $F_1(x, y, z) = F_1(\xi, \eta, \zeta)$ and $F_2(x, y, z) = F_2(\xi, \eta, \zeta)$ respectively, these surfaces being the members of the families (58) which pass through P. From this fact flows the following theorem about curves:

* A congruence

$$\left\| \begin{matrix} dx & dy & dz \\ X & Y & Z \end{matrix} \right\| = 0$$

is a *normal congruence* when there exists a one-parameter family of surfaces which cut orthogonally the curves of the congruence. The necessary and sufficient condition for this is that the equation

$$X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) = 0$$

is identically satisfied.

THEOREM 12. *In the portion of the region* Σ *for which* $U^2+V^2+W^2>0$, *a point* $P=(\xi,\eta,\zeta)$ *of a curve* c *is, or is not, a brilliant point of* c *with respect to the two congruences* (2), *according as* c *is, or is not, tangent at* P *to at least one of the two surfaces* $F_i(x,y,z)=F_i(\xi,\eta,\zeta),(i=1,2)$. *Furthermore if* c *is tangent at* P *to only one of the surfaces* $F_i(x,y,z)=F_i(\xi,\eta,\zeta)$, $(i=1,2)$, P *is actual or virtual according as* c *is tangent to* $F_2(x,y,z)=F_2(\xi,\eta,\zeta)$ *or* $F_1(x,y,z)=F_1(\xi,\eta,\zeta)$; *and if* c *is tangent at* P *to both of these surfaces,* P *is an exceptional brilliant point.*

Let us now suppose that the congruence (57) is also normal and that the corresponding family of orthogonal surfaces is represented by the equation

$$\Psi(x,y,z)=C. \tag{59}$$

Then we have the following theorem about surfaces:

THEOREM 13. *In the portion of the region* Σ *for which* $U^2+V^2+W^2>0$, *a point* $P=(\xi,\eta,\zeta)$ *of a surface* t *is, or is not,* $\left\{\begin{smallmatrix}\text{a brilliant point}\\ \text{a second extended brilliant point}\end{smallmatrix}\right\}$ *of* t *with respect to the two congruences* (2), *according as* t *is, or is not, tangent at* P *to one of the* $\left\{\begin{smallmatrix}\text{two}\\ \text{three}\end{smallmatrix}\right\}$ *surfaces* $\left\{\begin{smallmatrix}F_i(x,y,z)=F_i(\xi,\eta,\zeta),\ (i=1,2)\\ F_i(x,y,z)=F_i(\xi,\eta,\zeta),\ (i=1,2),\ \Psi(x,y,z)=\Psi(\xi,\eta,\zeta)\end{smallmatrix}\right\}$. *Furthermore* P *is actual or virtual according as* t *is tangent to* $F_2(x,y,z)=F_2(\xi,\eta,\zeta)$ *or* $F_1(x,y,z)=F_1(\xi,\eta,\zeta)$.

REMARK. *All the facts stated in Theorems* 12 *and* 13 *are true, not only with respect to the two congruences* (2), *but also with respect to the two congruences* (47), (48), *for every choice of the functions* f_1 *and* f_2, *provided that the portion of* Σ *in which* $f_1^2-f_2^2=0$ *be also excluded.*

In the examples of §§ 5 and 8 the two one-parameter families of surfaces (58) are the ellipsoids and hyperboloids of revolution of which P_1 and P_2 are the common foci and P_1P_2 is the common axis, and the family (59) is the family of planes which pass through P_1P_2. These three families of surfaces form a triply orthogonal system. This is always true of the three families (58), (59), since the planes π_1 and π_2 are perpendicular and intersect in p.

§ 10. *Optical interpretations.*

In the example of § 5, the points P_1 and P_2 may be regarded as a point source of light and the eye of an observer respectively, and the right lines which pass through O, as reflecting wires of small gauge. The observer will then see as a bright surface the locus of the actual brilliant points. If the circles (κ) were reflecting wires, the observer would see as a bright surface the locus of the virtual brilliant points of the right lines which pass through O.

In general the curves of congruence (1) may be regarded as reflecting wires of small gauge, as may also the curves of the congruences (40) and (41). Under certain conditions the curves of congruences (2) may be regarded as rays of light.

The conditions under which the curves of a congruence

$$\frac{dx}{ds}=\bar{X},\qquad \frac{dy}{ds}=\bar{Y},\qquad \frac{dz}{ds}=\bar{Z},\tag{60}$$

may be regarded as rays of light in a medium of variable index of refraction $n(x, y, z)$ *are those under which the system of partial differential equations**

$$\begin{aligned}
(\bar{X}_1^2-1)\frac{\partial n}{\partial x}+\bar{X}\cdot\bar{Y}\cdot\frac{\partial n}{\partial y}+\bar{X}\cdot\bar{Z}\cdot\frac{\partial n}{\partial z}&=-n\left(\frac{\partial\bar{X}}{\partial x}\bar{X}+\frac{\partial\bar{X}}{\partial y}\bar{Y}+\frac{\partial\bar{X}}{\partial z}\bar{Z}\right)\\
\bar{Y}\bar{X}\frac{\partial n}{\partial x}+(\bar{Y}^2-1)\frac{\partial n}{\partial y}+\bar{Y}\cdot\bar{Z}\cdot\frac{\partial n}{\partial z}&=-n\left(\frac{\partial\bar{Y}}{\partial x}\bar{X}+\frac{\partial\bar{Y}}{\partial y}\bar{Y}+\frac{\partial\bar{Y}}{\partial z}\bar{Z}\right),\qquad(61)\\
\bar{Z}\bar{X}\frac{\partial n}{\partial x}+\bar{Z}\cdot\bar{Y}\cdot\frac{\partial n}{\partial y}+(\bar{Z}^2-1)\frac{\partial n}{\partial z}&=-n\left(\frac{\partial\bar{Z}}{\partial x}\bar{X}+\frac{\partial\bar{Z}}{\partial y}\bar{Y}+\frac{\partial\bar{Z}}{\partial z}\bar{Z}\right).
\end{aligned}$$

have a common solution.

The system of equations (61) is merely another form of the system of equations

$$\frac{d}{ds}\left(n\frac{dx}{ds}\right)-\frac{\partial n}{\partial x}=0,\quad \frac{d}{ds}\left(n\frac{dy}{ds}\right)-\frac{\partial n}{\partial y}=0,\quad \frac{d}{ds}\left(n\frac{dz}{ds}\right)-\frac{\partial n}{\partial z}=0,\tag{62}$$

which is obtained by minimizing the integral

$$I=\int_{(A)}^{(B)} n(x, y, z)\,ds.\dagger$$

When, in particular, congruence (60) is rectilinear, the coefficients of n in the right members of equations (61) are identically zero. When this is the case, we have the

THEOREM 14. *The necessary and sufficient condition that the lines of a rectilinear congruence may be regarded as rays of light, is that the congruence be a normal congruence.*

In order to prove this, let us denote by u, v, w the determinants of the matrix formed from the coefficients of left members of the last two of the equations (61). They are found to be:

$$u=\bar{X}\cdot\bar{X},\qquad v=\bar{X}\cdot\bar{Y},\qquad w=\bar{X}\cdot\bar{Z}.$$

* These three equations reduce to two, as is verified by multiplying in succession by $\bar{X}, \bar{Y}, \bar{Z}$, adding, and making use of the relations

$$\bar{X}^2+\bar{Y}^2+\bar{Z}^2=1,\qquad \bar{X}\frac{\partial}{\partial\xi}\bar{X}+\bar{Y}\frac{\partial}{\partial\xi}\bar{Y}+\bar{Z}\frac{\partial}{\partial\xi}\bar{Z}=0\qquad(\xi=x, y, z).$$

† It is assumed that n is positive for all values of x, y, z. See APPELL, *Traité de mécanique rationnelle*, vol. 1, §§ 146, 150; A. WINKELMANN, *Handbuch der Physik* (1893), vol. 2_1, p. 344.

Therefore by a well-known theorem * and identity,† the necessary and sufficient condition that there exists a function n which will make the left members of the last two of equations (61) vanish, is that $\overline{X}^2 D$ vanishes identically, where

$$D = \overline{X}\left(\frac{\partial \overline{Y}}{\partial z} - \frac{\partial \overline{Z}}{\partial y}\right) + \overline{Y}\left(\frac{\partial \overline{Z}}{\partial x} - \frac{\partial \overline{X}}{\partial z}\right) + \overline{Z}\left(\frac{\partial \overline{X}}{\partial y} - \frac{\partial \overline{Y}}{\partial x}\right).$$

For the other two pairs of the left members of equations (61) we find $\overline{Y}^2 \cdot D$ and $\overline{Z}^2 \cdot D$ instead of $\overline{X}^2 \cdot D$. Since not all three of the expressions $\overline{X}$, $\overline{Y}$, $\overline{Z}$ vanish at the same point, it follows that $D \equiv 0$. But this is the condition that congruence (60) be normal. Hence the theorem.

By means of a theorem due to LEVI-CIVITA ‡ a congruence which is normal and rectilinear may be refracted (or in particular reflected) into any other congruence which is normal and rectilinear. Therefore any normal rectilinear congruence may be refracted (or reflected) into a two-parameter family of radiating right lines. Therefore if the congruences (2) are normal and rectilinear, it is possible to find surfaces at which they may be refracted or reflected into families of radiating right lines. At the radiants of these two families, a point source of light and the eye of an observer may be placed. The curves of congruence (1) being regarded as reflecting wires of small gauge, the observer whose eye is at one of the radiants will see as a bright surface the locus of the actual brilliant points due to the source of light at the other radiant.

* THEOREM. *The necessary and sufficient condition that the two independent linear partial differential equations*

$$X_1(x, y, z)\frac{\partial f}{\partial x} + Y_1(x, y, z)\frac{\partial f}{\partial y} + Z_1(x, y, z)\frac{\partial f}{\partial z} = 0,$$

$$X_2(x, y, z)\frac{\partial f}{\partial x} + Y_2(x, y, z)\frac{\partial f}{\partial y} + Z_2(x, y, z)\frac{\partial f}{\partial z} = 0$$

have a common solution is that the equation

$$U\left(\frac{\partial V}{\partial z} - \frac{\partial W}{\partial y}\right) + V\left(\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z}\right) + W\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}\right) = 0,$$

n w'ie'

$$U = \begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, \qquad V = \begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix}, \qquad W = \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix},$$

entically satisfied

† Identity:

$$\Theta X\left(\frac{\partial(\Theta Y)}{\partial z} - \frac{\partial(\Theta Z)}{\partial y}\right) + \Theta Y\left(\frac{\partial(\Theta Z)}{\partial x} - \frac{\partial(\Theta X)}{\partial z}\right) + \Theta Z\left(\frac{\partial(\Theta X)}{\partial y} - \frac{\partial(\Theta Y)}{\partial x}\right)$$
$$= \Theta^2\left[X\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + Y\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + Z\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)\right].$$

‡ THEOREM. *Two rectilinear congruences (both normal or both non-normal) are always obtainable one from the other by means of a finite number of refractions. More precisely, for the normal congruences one refraction suffices, for the others there occur in general two. The indices of refraction may be taken at pleasure, in particular equal to* -1, *which corresponds to reflection.*

See R. Accademia dei Lincei, *Rendiconti*, ser. 5, vol. 9[1] (1900), pp. 185 and 237.

Another kind of optical interpretation may also be given to the example of § 5 and, under certain broad restrictions, to the general problem of § 2. It is evident from Theorem 12 that the surface (α) of § 5 may be regarded as the locus of the lines of shade of the ellipsoids and hyperboloids of revolution of which P_1 and P_2 are the common foci, with regard to the rays of light which emanate from a point source at O. The portion of the surface (α) which is the locus of the actual brilliant points is the locus of the lines of shade of the ellipsoids, and that which is the locus of the virtual brilliant points is the locus of the lines of shade of the hyperboloids. Thus we deduce

A property of surface (α). The surface (α), § 5, has a generation of conics, the conic-generator being the intersection of the quadric surface of revolution of which P_1 and P_2 are the foci with the polar plane of this quadric surface with respect to the origin O. These conic-generators are in general not circles.

If the congruences (56) are normal, surfaces exist for the general problem of § 2 which play the rôle of the ellipsoids and hyperboloids in the example of § 5. These are the surfaces (58). The locus of the brilliant points of congruence (1) with respect to the two congruences (2) may then be regarded as the locus of the lines of shade of the surfaces (58) with regard to the congruence (1), or any of the congruences (40) or (41), for every choice of the functions m and n.

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CONTINUOUS INCREASING FUNCTIONS OF FINITE AND TRANSFINITE ORDINALS*

BY

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Introduction.†

A continuous increasing function of a set of ordinal numbers is analogous to a progressively continuous increasing function of the real variable. Some of its properties‡ are developed below, especially such as bear on the notion of a derived function of the αth degree (cf. § 3) and its extensions (cf. § 4). They are nearly all generalizations of properties discovered by CANTOR§ for particular functions and so may be used to simplify some of his proofs and generalize some of his results. In particular they extend his theory of ϵ-numbers.

One of the most interesting problems in the theory of transfinite numbers arises in connection with HARDY'S‖ scheme for obtaining a subset of the continuum of type Ω. The success of his method depends on determining for each ordinal number α ($\alpha \neq \beta + 1$) of the second class a *unique* fundamental sequence $S_\alpha = \{\alpha_\nu\}$ such that $L\alpha_\nu = \alpha$. For each number α ($\alpha \neq \beta + 1$) of the second class there evidently exists an infinitude of such sequences, of which, in any special case, one may be selected. But no one has as yet given a method of determining a set of sequences $\{S_\alpha\}$ such that for each α ($\alpha \neq \beta + 1$, $\omega \leqq \alpha < \Omega$) there exists one and only one S_α.

If α is restricted to be less than ϵ_0, the first ϵ-number, the problem of determining $\{S_\alpha\}$ is very easily solved. For every number of the second kind¶ in the second number-class can be written uniquely in the form,**

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† The rest of the article can be read independently of the introduction.

‡ Many others could be transferred to this theory from the theory of the functions of a real variable, for example, the theorem that a continuous function of a continuous function is continuous.

§ G. CANTOR, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 49 (1897).

‖ G. H. HARDY. Quarterly Journal of Mathematics, Vol. 35 (1903), pp. 87–94.

¶ CANTOR, loc. cit., p. 226.

** CANTOR, loc. cit., p. 237, Theorem B.

$$\alpha = \beta + \omega^{\gamma} \qquad (\gamma \neq 0),$$

in which γ denotes what CANTOR calls the exponent of α.

Now take $\{\nu\}$ (ν finite) as the fundamental sequence for ω; if γ is of the first kind, take $\{\beta + \omega^{\gamma-1}\nu\}$ (ν finite) as the fundamental sequence for $\beta + \omega^{\gamma}$; and if γ is of the second kind, take $\{\beta + \omega^{\gamma_\nu}\}$ (where $\{\gamma_\nu\}$ is the fundamental sequence for γ) as the fundamental sequence for $\beta + \omega^{\gamma}$. Since $\omega^{\alpha} > \alpha$ for all α's less than ϵ_0, this method determines $\{S_\alpha\}$ for all such α's, but since $\omega^{\epsilon_0} = \epsilon_0$ it fails to determine a sequence for ϵ_0. A method for extending this determination of $\{S_\alpha\}$ much further, though not so as to include all ordinals less than Ω, is given in Example 6.

§ 1. Let $\{x\}$ be the set of all finite and transfinite ordinal numbers less than a given ordinal X.

A *continuous increasing function* f is subject to the following conditions:

1) For every x of $\{x\}$, $f(x)$ is an ordinal number.

2) If $x_1 < x_2$, $f(x_1) < f(x_2)$.

3) If $\{x'\}$ is the set of all ordinals less than an ordinal x of the second kind, then $f(x)$ is the least ordinal greater than every $f(x')$.*

A set of ordinals is said to be *internally closed* if it includes all its limit-values with the possible exception of its least upper bound. On account of condition 3) the set of values y satisfying the conditions $y = f(x)$ is internally closed. Conversely, let $\{y\}$ be any internally closed set of ordinals. Then $\{y\}$ is a well-ordered (or *eutactic*) set and hence is similar to the set of all ordinals less than a certain X. Calling the latter set $\{x\}$ and denoting the ordered correspondence between $\{x\}$ and $\{y\}$ by f, it is evident that the function f satisfies the conditions 1), 2), 3). Hence,

THEOREM 1. *The set of values of $f(x)$, if f is any continuous increasing function, is internally closed. Conversely, any closed set of ordinals defines by its correspondence with a segment of the set of ordinals a continuous increasing function.*

EXAMPLE 1. As a function of β, $\alpha + \beta$ is continuous and increasing, but as a function of α, it is discontinuous.

EXAMPLE 2. The product $\alpha\beta$ is a continuous increasing function of β but not of α.

EXAMPLE 3. The numbers of the second kind of any number class and all preceding classes form an internally closed set. They therefore define a continuous increasing function $S(x)$.

* This may be expressed by the notation

$$\overline{B}_{x' \leq x} \{f(x')\} = f(x)$$

and the language "$f(x)$ is the least upper bound of the set $\{f(x')\}$."

EXAMPLE 4. If x_α is any element of a well-ordered set $\{x\}$, then the set of all elements preceding x_α is called a *section* ("Abschnitt") of $\{x\}$ and the set consisting of x_α and all following elements is called a *residue* ("Rest") of $\{x\}$.† A well-ordered set is never similar to a section of itself, but some well-ordered sets are similar to all their residues.

Such a set is called *self-residual* and its type or ordinal number is also called self-residual. An equivalent definition is that a self-residual number β satisfies the equation

$$\alpha + \beta = \beta$$

for every α less than β. The smallest self-residual number is evidently ω. Also if $\{\alpha\}$ is any set of self-residual numbers, $\bar{B}\{\alpha\}$ is easily seen to be self-residual because, if it had a residue less than itself, this residue as well as the corresponding section would be less than one of the α's.

Now let $X > \omega$ be the first ordinal of some number class and consider all the self-residual numbers of the set $\{x\} < X$. Since the set of self-residual numbers in $\{x\}$ has just been seen to be internally closed it defines a continuous increasing function which we shall denote by $r(x)$.

THEOREM 2. *There exists one and but one continuous increasing function $f(x)$ with a given value for $f(1)$ if the value of $f(x+1)$ is given uniquely for each value of $f(x)$.*

Proof. Suppose the theorem not valid; then there must be a first value x_0 for which $f(x)$ is not uniquely defined. If x_0 is of the first kind, there exists a number x_0' such that $x_0' + 1 = x_0$ and $f(x_0')$ is uniquely defined, and therefore $f(x_0)$ also, contrary to the hypothesis. If x_0 is of the second kind, it is the least ordinal larger than all those smaller ordinals for which $f(x)$ is uniquely defined. Hence by Condition 3) $f(x_0)$ must be the least upper bound of the set of values $f(x)$ for $x < x_0$. Thus $f(x_0)$ is again uniquely defined, contrary to hypothesis.

Corollary 1. The function $S(x)$ in Example 3 above is the same as $\omega \cdot x$.

Proof. If $S(x)$ is any number of the second kind, it is well known that $S(x+1)$, the next following ordinal of the second kind, is $S(x) + \omega$. Also it is well known that

$$\omega \cdot (x+1) = \omega \cdot x + \omega.$$

By definition,

$$\omega = \omega \cdot 1 = S(1).$$

As $\omega \cdot x$ and $S(x)$ are both continuous increasing functions, Theorem 2 gives

$$\omega \cdot x = S(x).$$

† CANTOR, loc. cit., p. 210.

EXAMPLE 5. According to CANTOR's definition † when translated into our terminology, α^x is the one continuous increasing function for which

$$\alpha^1 = \alpha, \qquad \alpha^{x+1} = \alpha^x \cdot \alpha.$$

Corollary 2. The self-residual function $r(x)$ defined in Example 4 is the same as ω^x.

Proof. If x is self-residual, $x + \alpha$ ($\alpha \leqq x$) can not be self-residual because one of its residues is α. Hence the self-residual ordinal next larger than x is $x \cdot \omega$. Since $r(1) = \omega$, this requires by Theorem 2 that

$$r(x) = \omega^x.$$

THEOREM 3. *A continuous increasing function f always satisfies the condition*

$$f(x) \geqq x.$$

Proof. Supposing the theorem untrue, let x_0 be the first value for which $f(x) < x$. Since $f(1) \geqq 1$, $x_0 > 1$. Hence if x_0 were of the first kind, $f(x_0') \geqq x_0'$ where $x_0' + 1 = x_0$, and so by Condition 2) $f(x_0) \geqq x_0$, which gives a contradiction. If x_0 were of the second kind, for all preceding values of x we should have $f(x) \geqq x$. But if so,

$$\overline{B}_{x<x_0}\{f(x)\} \geqq \overline{B}_{x<x_0}\{x\}$$

which is, by Condition 3), a contradiction of $f(x_0) < x_0$.

§ 2. Let us now introduce the conditions that $X > \omega$ shall be the first ordinal of a certain number class and that the values of $f(x)$ as well as of x shall be less than X. The latter condition is fulfilled, as may easily be shown, whenever $f(1) < X$ and the difference between $f(x)$ and $f(x+1)$ is always less than X. From here on, the functional symbols f, ϕ, etc., shall be used only for continuous increasing functions satisfying these additional conditions.

THEOREM 4. *There exist solutions of the equation $f(x) = x$ and these solutions $\{\xi\}$ form a closed set similar to $\{x\}$.*

Proof.‡ Let x_1, be any value of x. Consider the sequence

$$y_1 = f(x_1),\ y_2 = f(y_1),\ y_3 = f(y_2),\ \cdots,\ y_n = f(y_{n-1}),\ \cdots$$

for all finite values of n. The least upper bound ξ of the arguments y_{n-1} is also the least upper bound of the functional values, y_n. Hence, by Condition 3) of the definition of a continuous increasing function, $f(\xi) = \xi$. This shows that for every value of x there is a larger or equal value of ξ. As $\{\xi\}$ is a subset of $\{x\}$ this requires that $\{\xi\}$ and $\{x\}$ shall be similar. To show that $\{\xi\}$ is closed,

† Loc. cit., p. 233.

‡ Cf. CANTOR's proof of Theorem A, loc. cit, p. 242.

let $\{\xi'\}$ be any subset of $\{\xi\}$ whose least upper bound x_0 is of the second kind. Being identical with $\{\xi'\}$, the set $\{f(\xi')\}$ has also the upper bound x_0. Hence $f(x_0) = x_0$ and x_0 belongs to $\{\xi\}$.

The continuous increasing function defined by the correspondence between $\{x\}$ and $\{\xi\}$ is called the *first derived function of* f.

Corollary 1. If f' is the first derived function of f, $f'(1)$ is the least upper bound of $f(1), f[f(1)], \cdots$, and if $f'(x) < \alpha < f'(x+1)$, then $f'(x+1)$ is the least upper bound of $f(\alpha), f[f(\alpha)], \cdots$.

Corollary 2. The first derived function of $\alpha + x$ is $\alpha \cdot \omega + (x-1)$.†

Corollary 3. The first derived function of $\alpha \cdot x$ is $\alpha^\omega \cdot x$.

Proof. By Corollary 1, α^ω is the first ordinal which satisfies

$$\alpha \cdot x = x. \tag{1}$$

Also if ξ satisfies (1),

$$\alpha(\xi + 1) = \alpha\xi + \alpha = \xi + \alpha,$$

and hence by Corollary 1, $\xi + \alpha^\omega$ is the next ordinal satisfying (1). According to Theorem 2, this shows that $\alpha^\omega \cdot x$ is identical with the first derived function of $\alpha \cdot x$.

Corollary 4. The first derived function of ω^x is the function ϵ, where $\epsilon(x)$ stands for the ϵ-number ϵ_{x-1}.‡

Corollary 5. The first derived function of α^x $(\alpha > \omega)$ is the function f, where $f(x)$ stands for the ϵ-number $\epsilon_{\beta+(x-1)}$ if ϵ_β is the first ϵ-number larger than α. The first derived function of n^x (n finite) is the function ϕ, where $\phi(1) = \omega$ and $\phi(x) = \epsilon_{x-2}$ when $1 < x$.§

THEOREM 5.‖ *If $\{S_\alpha\}$ is a well-ordered set of internally closed sets S_α of ordinals, each S_α being a subset of $\{x\}$ similar to $\{x\}$ and also a subset of each S_α which precedes it, then, provided the cardinal number $\overline{\{S_\alpha\}}$ is less than $\overline{\{x\}}$, there exists an internally closed set S which includes all the ordinals common to all S_α's and is similar to $\{x\}$.*

Proof. Let x be any ordinal of $\{x\}$ and y_α be the first ordinal of S_α which is larger than x. The set $\{y_\alpha\}$ is, by hypothesis, of cardinal number less than $\overline{\{x\}}$. Hence as $\{y_\alpha\}$ is a subset of $\{x\}$ it has a least upper bound y in $\{x\}$.

Since $\{y_\alpha\}$ is a subset of S_1, y is a member of S_1. In like manner, since the set of all y_α's for which $\alpha > \beta$ is a subset of S_β, y is a member of S_β. Hence y is a member of every S_α. Thus the set S of all ordinals y common to the sets S_α exists and is similar to $\{x\}$. That S is internally closed follows from the fact that any subset S' of S whose least upper bound is not X is a subset of each S_α

† Here it is to be remembered that if $x > \omega$, $x - 1 = x$.

‡ CANTOR, loc. cit., § 20.

§ CANTOR, theorems *G* and *H*, p. 245.

‖ For the notation cf. CANTOR, Mathematische Annalen, vol. 46 (1895), pp. 481–2.

and hence as each S_α is internally closed the least upper bound of S' is a member of each S_α.

THEOREM 6. *For every continuous increasing function $f(x)$ there exists a well-ordered set of continuous increasing functions $f(x, \alpha)$, $\alpha < X$, such that the values of $f(x, \alpha)$ (for fixed α and variable x) are the set of all ordinals which are solutions of*

$$f(x, \beta) = x,$$

for every β less than α.

Proof. In view of Theorems 4 and 1 this is a corollary of Theorem 5. The first derived function of $f(x, \alpha)$ is $f(x, \alpha + 1)$, and $f(x, \alpha)$ may be called the *αth derived function* of $f(x)$.

Corollary 1. If $f(1) > 1$ the function $f(1, \alpha)$ is a continuous increasing function of α, α taking all values of $\{x\}$.

Proof. Since $f(1) > 1$, we have $f(1, 1) > f(1)$ by Condition 2) of the definition of a continuous increasing function taken with Corollary 1 of Theorem 4. In like manner $f(1, \alpha + 1) > f(1, \alpha)$, and hence Conditions 1) and 2) of the definition of a continuous increasing function are satisfied by $f(1, \alpha)$. To show that Condition 3) is satisfied, consider any set of α's, $\{\alpha'\}$, whose least upper bound β is not X. $\bar{B}\{f(1, \alpha')\}$ is a value of $f(x, \alpha_0')$ for each α_0' of $\{\alpha'\}$ because every value of $f(1, \alpha')$ for $\alpha' > \alpha_0'$ is a value of $f(x, \alpha_0')$. Hence as $f(1, \beta)$ cannot be less than $\bar{B}\{f(1, \alpha')\}$ it must be equal to it.

Corollary 2. If $f(x) = \gamma + x$ (γ constant),

$$f(x, \alpha) = \gamma \cdot \omega^\alpha + (x - 1),$$

and so $\gamma \cdot \omega^\alpha$ is a continuous increasing function of α.

Corollary 3. If $f(x) = \gamma \cdot x$ (γ constant),

$$f(x, \alpha) = \gamma^{\omega^\alpha} \cdot x,$$

and hence the function γ^{ω^α} is a continuous increasing function of α.

Corollary 4. If $f(x) = \omega^x$, $f(x, \alpha)$ is the $(\alpha - 1)$th derived function of $\epsilon(x)$. For each α, $\{f(x, \alpha)\}$ is an internally closed set of ϵ-numbers. Also $\{f(1, \alpha)\}$ is an internally closed set of ϵ-numbers.

§ 3. In order to extend the notion of a derived function still further, let us consider the set of symbols

$$\{\phi(x_1, x_2, \cdots, x_\beta)\} = \{\phi\},$$

where β may be any ordinal less than X and $x_\alpha (\alpha = 1, 2, \cdots, \omega_1, \cdots, \beta)$ may be any ordinal less than X, though in a given symbol ϕ only a finite number of values x_α are different from 1. We shall refer to a symbol $\phi(x_1, x_2, \cdots, x_\beta)$

as of βth *order*. Let the set $\{\phi\}$ be ordered according to last differences,† i. e., as follows: Of two symbols

$$\phi_1 = \phi(x_1, x_2, \cdots, x_\beta), \qquad \phi_2 = \phi(y_1, y_2, \cdots, y_\gamma),$$

ϕ_1 precedes ϕ_2 if $\beta < \gamma$, or if $\beta = \gamma$ and $x_\beta < y_\gamma$, or if $\beta = \gamma$ and $x_\alpha < y_\alpha$ for some α ($\alpha < \beta$) while $x_\rho = y_\rho$ for all values of ρ such that $\alpha < \rho \leqq \beta$.

According to this rule $\{\phi\}$ is simply ordered and it may be shown to be well ordered by proving that every subset $\{\phi'\}$ has a first element. Let β_1 be the smallest final subscript in any element of $\{\phi'\}$. The elements of $\{\phi\}$ of order β_1 form a set $\{\phi''\}$ and precede all other elements of $\{\phi'\}$. Let y_{β_1} be the smallest value of x_{β_1} in any element of $\{\phi''\}$.

If there is more than one element of this kind, they form a set $\{\phi_1\}$ of elements which precede all other elements of $\{\phi'\}$. Let β_2 be the lowest subscript in any element $\bar{\phi}$ of $\{\phi_1\}$ such that in $\bar{\phi}$ nothing but 1's appear between x_{β_2} and y_{β_1}, and let y_{β_2} be the smallest x_{β_2} in any element $\bar{\phi}$. This determines a set of elements $\{\phi_2\}$ which precede all other elements of $\{\phi'\}$. If $\{\phi_2\}$ contains more than one element, repeat this process, obtaining sets $\{\phi_3\}, \{\phi_4\}, \cdots, \{\phi_n\}$ where ϕ_κ consists of elements of the type

$$\phi(x_1, \cdots, y_{\beta_\kappa}, 1_{\beta_\kappa+1}, \cdots, y_{\beta_{\kappa-1}}, 1_{\beta_{\kappa-1}+1}, \cdots, y_{\beta_1}).$$

By a finite number of steps this must lead to a set $\{\phi_m\}$ consisting of only one element ϕ_m, because a sequence $\beta_1, \beta_2, \beta_3, \cdots$ of type ω^* is impossible. Thus ϕ_m is the required first elements of $\{\phi'\}$.

The symbols of $\{\phi\}$, aside from $\phi(1)$ the first element, fall into seven classes.‡

A. $\phi(1_1, 1_2, \cdots, 1_\alpha)$ (α of 1st kind)

$$= \bar{B}\{\phi(1_1, 1_2, \cdots, x_{\alpha'})\} \text{ for all values of } x_{\alpha'} \text{ where } \alpha' + 1 = \alpha.$$

B. $\phi(1_1, 1_2, \cdots, 1_\alpha)$ (α of 2d kind)

$$= \bar{B}\{\phi(1_1, 1_2, \cdots, 1_\gamma)\} \text{ for } \gamma < \alpha.$$

C. $\phi(x_1, \cdots, x_\beta)$ ($x_1 > 1$, 1st kind) has a next preceding element $\phi(x_1', \cdots, x_\beta)$, where $x_1' + 1 = x_1$.

D. $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$ ($\alpha > 1$, 1st kind, x_α of 1st kind)

$$= \bar{B}\{\phi(1_1, \cdots, x_{\alpha'}, x_\alpha', \cdots, x_\beta)\}$$

† The ordering of various sets analogous to $\{\phi\}$ according to first or last differences ha been studied by F. HAUSDORFF; *Untersuchungen über Ordnungstypen*, Leipziger Berichte, 1906–07.

‡ In this notation x_α indicates the first non-unity ordinal in the symbol

$$\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$$

if such an ordinal exists.

for all $x_{\alpha'} < X$ where $x'_\alpha + 1 = x_\alpha$ and $\alpha' + 1 = \alpha$.

E. $\phi(1_1, \cdots, x_\alpha, x_{\alpha+1}, \cdots, x_\beta)$ ($\alpha \geqq 1$, 1st kind, x_α of 2d kind)

$= \bar{B}\{\phi(1_1, \cdots, x'_\alpha, x_{\alpha+1}, \cdots, x_\beta)\}$ for all $x'_\alpha < x_\alpha$.

F. $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$ (α of 2d kind, x_α of 1st kind)

$= \bar{B}\{\phi(1_1, \cdots, 2_\gamma, \cdots, x'_\alpha, \cdots, x_\beta)\}$ for all $\gamma < \alpha$ where $x'_\alpha + 1 = x_\alpha$.

G. $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$ (α of 2d kind, x_α of 2d kind)

$= \bar{B}\{\phi(1_1, \cdots, x'_\alpha, \cdots, x_\beta)\}$ for all $x'_\alpha < x_\alpha$.

Note that the elements of $\{\phi\}$ of the first kind are $\phi(1)$ and the class C. The elements of the second kind which are least upper bounds of sequences of type X are classes A and D; and those which are least upper bounds of sequences which are sections of X are classes B, E, F, and G. All the elements of classes A and B form an internally closed set of order-type X. Again all symbols of type

$$\phi(1_1, 1_2, \cdots, x_\alpha, \cdots, x_\beta),$$

where all the digits except x_α are fixed, form an internally closed set. It may happen that all the elements of one internally closed set come between two elements of some other set, as

$$\{\phi(x_1, 1_2, \cdots, x_\alpha, \cdots, x_\beta)\}$$

come between two elements of

$$\{\phi(1_1, 1_2, \cdots, x_\alpha, \cdots, x_\beta\}.$$

§ 4. Let $\phi(x_1)$ be any continuous increasing function, i. e., let $\{\phi(x_1)\}$ be an internally closed subset of $\{x_1\}$. Then by $\phi(x_1, 1_2, 1_3, \cdots, 1_\beta)$ is meant the set of all common solutions of the equations

$$\phi(1_1, 1_2, \cdots, x_\gamma) = x_\gamma \tag{1}$$

for $\gamma < \beta$, and by $\phi(x_1, 1_2, 1_3, \cdots, x_\alpha, \cdots, x_\beta)$, $\alpha > 1$, is meant the set of all common solutions of the equations

$$\phi(1_1, 1_2, \cdots, x_\gamma, 1_{\gamma+1}, \cdots, x'_\alpha, \cdots, x_\beta) = x_\gamma \tag{2}$$

for $\gamma < \alpha$ and $x'_\alpha < x_\alpha$, the numbers x_ρ in the symbol being constants with the exception of x_γ and x'_α.

Note that this coincides with the definition of derived function in § 2 for the cases $\phi(x_1, x_2)$. As an extension of Theorem 6, we have

THEOREM 7. *If $\phi(x_1)$ is an arbitrary continuous increasing function for which $\phi(1) > 1$, this definition determines one and only one value of $x < X$ for each symbol $\phi(x_1, \cdots, x_\beta)$.*

Proof. (*a*). We first show that if the theorem is true for all symbols preceding a certain one,

$$\phi(1_1, \cdots, \bar{x}_\alpha, \bar{x}_{\alpha+1}, \cdots, \bar{x}_\beta) = \bar{\phi},$$

and if $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta) = \delta(x_\alpha)$ is any function of x_α (all ordinals before x_α in the symbol being 1, as usual) the symbols for all of whose values precede $\bar{\phi}$, then $\delta(x_\alpha)$ is a continuous increasing function.

If this were not so, let $\phi(1_1, \cdots, x_\gamma, \cdots, x_\delta)$ be the first element which appeared in a function $\delta(x_\alpha)$ which was not continuous and increasing. The first non-continuous increasing function in which this element could appear would be

$$\phi(1_1, \cdots, x_\epsilon, \cdots, x_\gamma, \cdots, x_\delta) = \delta(x_\epsilon) \qquad (\epsilon \leqq \gamma).$$

Since the sets of equations (1) and (2) in the definitions are well-ordered and of cardinal less than $\{x\}$, if $\epsilon = 1$, the function $\delta(x_\epsilon)$ would have to be continuous and increasing on account of Theorems 4 and 5.

But if $\epsilon > 1$, the definition shows that when x_ϵ is of the first kind and x'_ϵ is its next preceding ordinal,

$$\phi(1_1, \cdots, x'_\epsilon, \cdots, x_\gamma, \cdots, x_\delta) < \phi(2_1, 1_2, \cdots, x'_\epsilon, \cdots, x_\gamma, \cdots, x_\delta) < \phi(1_1, \cdots, x_\epsilon, \cdots, x_\gamma, \cdots, x^\delta),$$

and hence $\delta(x_\epsilon)$ satisfies Condition 2) of the definition of a continuous increasing function. The definition also shows that when x_ϵ is of the second kind $\delta(x_\epsilon) = \bar{B}\, \delta(x'_\epsilon)$ for $x'_\epsilon < x_\epsilon$ and hence $\delta(x_\epsilon)$ satisfies Condition 3) as well.

Condition 1) is evidently satisfied. Hence $\delta(x_\epsilon)$ is always a continuous increasing function, contrary to hypothesis.

(*b*) Now supposing our theorem untrue, let $\phi(x_1, \cdots, x_\beta) = \Phi$ be the first symbol for which there is not determined a unique value. Φ cannot belong to classes A or B (cf. § 3) because then the first part of our definition determines for it a value which exists and is unique by Theorems 4 and 5 together with paragraph (*a*) of this proof. In like manner Φ cannot belong to class C or to class E with $\alpha = 1$ because when $x_2 = 1, \cdots, x_\beta = 1$ its value is determined by first part of our definition and in the other cases by the second part. Φ cannot belong to the classes D, E, F, or G because in these cases its value is determined by the second part of the definition.

Corollary. $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$ is a continuous increasing function of x_α.

THEOREM 8. $\phi(1_1, \cdots, x_\alpha, \cdots, x_\gamma, \cdots, x_\beta) > x_\gamma$ *if* $x_\alpha > 1$.

Proof. By Theorem 3 and the corollary above,

$$\phi(1_1, \cdots, 1_\alpha, \cdots, x_\gamma, \cdots, x_\beta) \geqq x_\gamma,$$

and for a given x_γ

$$\phi(1_1, \cdots, x_\alpha, \cdots, x_\gamma, \cdots, x_\beta) > \phi(1_1, \cdots, 1_\alpha, \cdots, x_\gamma, \cdots, x_\beta).$$

Corollary. $\phi(1_1, \cdots, x_\alpha, \cdots, x_\gamma, \cdots, x_\beta)$ is not a continuous increasing function of x_γ if $x_\alpha > 1$.

THEOREM 9. $\phi(1_1, \cdots, 1_\beta)$ *is a continuous increasing function of* β.

This is evident from the definition of this function. On the other hand $f(1_1, \cdots, 2_\beta)$ is not a continuous increasing function because on account of the inequality

$$f(1_1, \cdots, 2_\beta) > f(1_1, \cdots, 1_\beta) \geqq \beta$$

it cannot satisfy $f(1_1, \cdots, 2_\beta) = \beta$.

This new continuous increasing function

$$f(1_1, \cdots, 1_\beta) = \psi(\beta)$$

can be used to generate a new set of functions

$$\psi(1_1, \cdots, x_\alpha, \cdots, x_\beta),$$

and the process continued indefinitely.

If from the set S of functional symbols $\phi(x_1, \cdots, x_\beta)$ is omitted each symbol whose value is equal to a value of its first derived function, there remains a set of functional symbols which we denote by T.

THEOREM 10. *The ordinal number represented by each symbol in* T *satisfies the condition* .

$$\phi(x_1, \cdots, x_\gamma, \cdots, x_\beta) > x_\gamma \qquad (1 \leqq \gamma \leqq \beta).$$

Proof. Let x_α be the first non-unity element in $\phi(x_1, \cdots, x_\beta)$ which is now to be written $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$. Then the equation

$$\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta) = x_\alpha$$

is impossible because in that case $\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$ would be equal to $\phi(x_1, \cdots, 1_\alpha, x_{\alpha+1} + 1, \cdots, x_\beta)$ and thus could not be a member of T. Hence by Theorem 3

$$\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta) > x_\alpha.$$

Now let x_δ be the first non-unity element after x_α. It follows that

$$\phi(1_1, \cdots, x_\alpha, \cdots, x_\delta, \cdots, x_\beta) > \phi(1_1, \cdots, 1_\alpha, \cdots, x_\delta, \cdots, x_\beta),$$

and by the argument above

$$\phi(1_1, \cdots, 1_\alpha, \cdots, x_\delta, \cdots, x_\beta) > x_\delta.$$

As there are only a finite number of non-unity elements between x_α and x_β a finite number of repetitions of this step establishes the theorem.

EXAMPLE 6. Let $\phi(x_1) = 1 + x_1$ and form the set S of functions $\phi(x_1, \cdots, x_\beta)$. Then

$\phi(x_1, 1_2) = \omega + (x_1 - 1)$ (cf. Corollary 2, Theorem 4).

$\phi(x_1, 2_2) = \omega^2 + (x_1 - 1)$

. . . . . .

$\phi(x_1, x_2) = \omega^{x_2} + (x_1 - 1)$ (cf. Corollary 2, Theorem 6).

. . . . . .

Hence $\phi(1_1, x_2) = \omega^{x_2}$.

$\phi(1_1, 1_2, 1_3) = \epsilon(1) =$ the smallest ϵ-number (cf. Corollary 4, Theorem 4).

$\phi(x_1, 1_2, 1_3) = \epsilon(x_1)$, which are the ϵ-numbers.

. . . . . .

$\phi(1_1, 2_2, 1_3)$ is the smallest ϵ-number equal to its subscript.

$\phi(x_1, x_2, 1_3)$ are the generalization of ϵ-numbers of Corollary 4, Theorem 6.

. . . . .

$\phi(x_1, \cdots, x_\alpha)$ are further generalizations of ϵ-numbers. By fixing all the x_ρ's except one which is preceded only by 1's we obtain an internally closed subset of ϵ-numbers.

The set T based on the function $1 + x$ gives expressions in terms of smaller ordinals in the form:

$\phi(x_1) = 1 + x_1$ for ordinals greater than or equal to 1 and less than ω,

$\phi(x_1, 1_2) = \omega + (x_1 - 1)$ for ordinals greater than or equal to ω and less than ω^2,

$\phi(x_1, x_2) = \omega^{x_2} + (x_1 - 1)$ for ordinals greater than or equal to ω^{x_2} and less than ω^{x_2+1}, with the exceptions of those ordinals which satisfy the condition $\phi(1, x_2) = x_2 [= \omega^{x_2}]$. These are expressed by means of

$\phi(x_1, 1_2, 1_3)$ [their ranks as ϵ-numbers] unless they are values of

$\phi(x_1, 2_2, 1_3)$

. . . . .

Ordinals not denoted by any previous symbol are expressed in terms of

$\phi(1_1, \cdots, x_\alpha, \cdots, x_\beta)$ unless they satisfy the equation

$\phi(1, \cdots, x_\alpha, \cdots, x_\beta) = x_\alpha$, in which case they are expressed by means of

$\phi(x_1, \cdots, x_\alpha + 1, \cdots, x_\beta)$ or some later symbol.

By this scheme every ordinal is expressed in terms of smaller ordinals and *by means of a symbol involving subscripts smaller than itself* with the exception of those numbers which satisfy the equation

$$\phi(1_1, 1_2, \cdots, 1_\alpha, \cdots, 1_\beta) = \beta.$$

The solutions of this equation are clearly an internally closed set and define a continuous increasing function

$$\mathrm{E}(x)$$

and may be called the E-numbers. Of course this function $\mathrm{E}(x)$ may be made the basis of a new set T' and the expression of ordinals in terms of smaller ordinals continued indefinitely.

We may now solve the problem of determining for each ordinal of the second kind and less than $\mathrm{E}(1)$ a unique sequence of type ω of which it is the least upper bound. Any such ordinal is expressed in terms of smaller ordinals in the form

$$\phi(x_1, \cdots. x_\alpha, \cdots, x_\beta) = \Phi \qquad (\alpha \geqq 1),$$

which belongs to one of the classes $A - G$ of § 3.

If Φ is of class A, its sequence is $\{\xi^\nu\}$ where

$$\xi' = \phi(1_1, 1_2, \cdots, 1_{\alpha'}), \cdots, \xi^\nu = \phi(1_1, 1_2, \cdots, \xi^{\nu-1}_{\alpha'}).$$

If Φ is of class B, its sequence is $\{\phi(1_1, 1_2, \cdots, 1_{\gamma_\nu})\}$ where $\{\gamma_\nu\}$ is the sequence for α.

If $\Phi = \phi(x_1, \cdots, x_\beta)$ is of class C, let $\phi(x_1', \cdots, x_\beta) = \rho$ where $x_1' + 1 = x_1$, and let Φ' be the element obtained by replacing x_1 by 1 in Φ. The element Φ' must belong to one of the classes A, B, D, E, F, G. If Φ' belongs to class A, then the sequence for Φ is $\{\xi^\nu\}$ where

$$\xi' = \rho + 1, \cdots, \xi^\nu = \phi(1_1, 1_2, \cdots, \xi^{\nu-1}_{\alpha'}), \cdots.$$

If Φ' belongs to class B, then the sequence for Φ is $\{\phi[1_1, 1_2, \cdots, (\rho+1)_{\gamma_\nu}]\}$ where $\{\gamma_\nu\}$ is the sequence for α.

If Φ' belongs to class D, then the sequence for Φ is $\{\xi^\nu\}$ where $\xi' = \rho + 1, \cdots, \xi^\nu = \phi(1_1, \cdots, \xi^{\nu-1}_{\alpha'}, x_\alpha', \cdots, x_\beta), \cdots$.

If Φ' belongs to class E or G then the sequence for Φ is $\{\phi[(\rho+1)_1, \cdots, x_\alpha^\nu, \cdots, x_\beta]\}$ where $\{x_\alpha^\nu\}$ is the sequence for x_α.

If Φ' belongs to class F, then the sequence for Φ is

$$\{\phi[(\rho+1)_1, \cdots, 2_{\gamma_\nu}, \cdots, x_\alpha', \cdots, x_\beta]\}$$

where $\{\gamma_\nu\}$ is the sequence for α.

If Φ is of classes D, E, F or G, unless it is of class E with $\alpha = 1$, its sequence is the same as the corresponding one developed above for class C, on setting $\rho = 0$.

If Φ is of class E with $\alpha = 1$, its sequence is $\{\phi(x_1^r, \cdots, x_\beta)\}$ where $\{x_1^r\}$ is the sequence for $x_1 = x_\alpha$.

This scheme serves to define a unique sequence for each ordinal smaller than $\mathrm{E}(1)$, but it would not serve for $\mathrm{E}(1)$ because the symbol for $\mathrm{E}(1)$ belongs to class B and involves a subscript equal to $\mathrm{E}(1)$ and therefore the sequence for $\mathrm{E}(1)$ would be defined in terms of $\mathrm{E}(1)$ itself. Of course the following procedure is possible. Let

$$\phi(1_1, 1_2, \cdots, 1_\beta) = f(\beta)$$

and form the set of functions

$$f(\beta_1, \cdots, \beta_\alpha)$$

and determine sequences for all the E-numbers less than the first solution of

$$f(1_1, \cdots, 1_\alpha) = \alpha$$

by means of the function

$$f(\beta_1, \cdots, \beta_\alpha),$$

and this may be repeated indefinitely.

PRINCETON, N. J.

PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(THIRD MEMOIR)

BY

E. J. WILCZYNSKI

§ 1. *Canonical development of the equation of a ruled surface.*

In the second memoir† it was shown that, in the vicinity of an ordinary point, the equation of any non-ruled surface could be expressed by a series of the form

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

where I, J and all of the following coefficients are absolute invariants of the surface. The geometrical significance of this development was fully brought out and gave rise to a number of new and important results. For a ruled surface, however, no series of this form exists. It is the purpose of this paper to find a canonical development for this exceptional case and to interpret it geometrically.

We shall assume that the partial differential equations of the surface are given in their canonical form

$$y_{uu} + 2by_v + fy = 0,$$

$$y_{vv} + 2a'y_u + gy = 0,$$

so that the curves $u = \text{const.}$ and $v = \text{const.}$ are the two families of asymptotic curves. The curves $u = \text{const.}$ will be straight lines, if and only if $a' = 0$.‡ This we shall assume from now on, so that our fundamental system of equations is

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + gy = 0. \tag{1}$$

These equations will characterize a ruled surface S whose generators are the curves $u = \text{const.}$ The integrability conditions of this system are §

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† These Transactions, vol. 9 (1908), p. 79.

§ First Memoir. These Transactions, vol. 8 (1907), p. 260.

‡ These Transactions, vol. 8 (1907), p. 247, eq. (34). In the last equation f and g should be interchanged.

$$(2) \qquad g_u = 0, \qquad b_{vv} + f_v = 0, \qquad -f_{vv} + 4gb_v + 2bg_v = 0.$$

They are supposed to be fulfilled so that, in particular, g will be a function of v only. The four fundamental semi-covariants are

$$(3) \qquad y, \qquad z = y_u, \qquad \rho = y_v, \qquad \sigma = y_{uv}.$$

Along a generator u remains constant, so that the line joining P_y to P_ρ is a generator of the surface and we may think of this surface as being the locus of the lines $P_y P_\rho$ which correspond to the various values of u.

For the purpose of our development we shall need the partial derivatives of y up to those of the fifth order. We may use the expressions for the derivatives of the first four orders which were obtained in the second memoir† and simplify them by putting $a' = 0$. This gives

$$(4) \quad y_u = z, \qquad y_v = \rho, \qquad y_{uu} = -fy - 2b\rho, \qquad y_{uv} = \sigma, \qquad y_{vv} = -gy,$$

and

$$(5) \quad \begin{aligned} &y_{uuu} = -f_u y - fz - 2b_u \rho - 2b\sigma, \qquad y_{uuv} = (2bg - f_v)y - (f + 2b_v)\rho, \\ &y_{uvv} = -gz, \qquad\qquad\qquad\qquad y_{vvv} = -g_v y - g\rho. \end{aligned}$$

We have further

$$(6) \quad \begin{aligned} &y_{uuuu} = \Sigma \alpha_1 y, \qquad y_{uuuv} = \Sigma \beta_1 y, \qquad y_{uuvv} = \Sigma \gamma_1 y, \\ &y_{uvvv} = \Sigma \delta_1 y, \qquad y_{vvvv} = \Sigma \epsilon_1 y, \end{aligned}$$

where the symbols on the right members are abbreviations for expressions of the form

$$\alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma, \text{ etc.},$$

and where

$$(7) \quad \begin{aligned} &\alpha_1 = f^2 - 4b^2 g + 2bf_v - f_{uu}, \quad \alpha_2 = -2f_u, \quad \alpha_3 = 2(2bf + 2bb_v - b_{uu}), \quad \alpha_4 = -4b_u, \\ &\beta_1 = 2b_u g - f_{uv}, \quad \beta_2 = 2bg - f_v, \quad \beta_3 = -(f_u + 2b_{uv}), \quad \beta_4 = -(f + 2b_v), \\ &\gamma_1 = fg, \quad \gamma_2 = 0, \quad \gamma_3 = 2bg, \quad \gamma_4 = 0, \\ &\delta_1 = 0, \quad \delta_2 = -g_v, \quad \delta_3 = 0, \quad \delta_4 = -g, \\ &\epsilon_1 = -g_{vv} + g^2, \quad \epsilon_2 = 0, \quad \epsilon_3 = -2g_v, \quad \epsilon_4 = 0. \end{aligned}$$

Similarly we may write

$$(8) \quad \begin{aligned} &y_{uuuuu} = \Sigma \alpha_1' y, \qquad y_{uuuuv} = \Sigma \beta_1' y, \qquad y_{uuuvv} = \Sigma \gamma_1' y, \\ &y_{uuvvv} = \Sigma \delta_1' y, \qquad y_{uvvvv} = \Sigma \epsilon_1' y, \qquad y_{vvvvv} = \Sigma \zeta_1' y, \end{aligned}$$

where

† These Transactions, vol. 9 (1908), pp. 98–99, eq. (78) to (80).

$$(9)\qquad \begin{aligned} &\alpha_4' = \frac{\partial \alpha_4}{\partial u} + \alpha_3, \qquad \beta_4' = \frac{\partial \alpha_4}{\partial v} + \alpha_2, \qquad \gamma_4' = \frac{\partial \beta_4}{\partial v} + \beta_2, \\ &\delta_4' = \frac{\partial \gamma_4}{\partial v} + \gamma_2, \qquad \epsilon_4' = \frac{\partial \delta_4}{\partial v} + \delta_2, \qquad \zeta_4' = \frac{\partial \epsilon_4}{\partial v} + \epsilon_2, \end{aligned}$$

or explicitly,

$$(10)\qquad \begin{aligned} &\alpha_4' = -6b_{uu} + 4bf + 4bb_v, \qquad \beta_4' = -4b_{uv} - 2f_u, \qquad \gamma_4' = 2bg, \\ &\delta_4' = 0, \qquad\qquad \epsilon_4' = -2g_v, \qquad\qquad \zeta_4' = 0, \end{aligned}$$

while the expressions of the other coefficients in (8) are not needed for our present purpose.

Let us consider the point $u = u_0$, $v = v_0$ of an integral surface of system (1). The values of y, y_u, y_v, etc., for this point can be computed by the formulæ just developed. Let Y or (u, v) represent a point in the vicinity of (u_0, v_0), and let us assume that the point (u_0, v_0) is an ordinary point of the surface, i. e., let us assume that the coefficients of (1) are regular about this point. Then we shall have by TAYLOR's development

$$Y = y + y_u(u - u_0) + y_v(v - v_0) + \cdots,$$

But, making use of (4) to (8), this may be put into the form

$$Y = x_1 y + x_2 z + x_3 \rho + x_4 \sigma,$$

where $x_1, \cdots, x_4$ are series proceeding according to positive integral powers of $u - u_0$ and $v - v_0$. As in previous papers we shall introduce $P_y P_z P_\rho P_\sigma$ as tetrahedron of reference in such a way that $(x_1, \cdots, x_4)$ become the homogeneous coördinates of the point Y with respect to it. Moreover, in place of $u - u_0$ and $v - v_0$ we shall write u and v respectively.

We obtain in this way the following series for the homogeneous coördinates of a point of the surface in the vicinity of P_y, the terms up to and including those of the fifth order being actually computed:

$$(11)\qquad \begin{aligned} x_1 &= 1 - \tfrac{1}{2}fu^2 - \tfrac{1}{2}gv^2 - \tfrac{1}{6}f_u u^3 + \tfrac{1}{2}(2bg - f_v)u^2 v - \tfrac{1}{6}g_v v^3 \\ &\quad + \tfrac{1}{24}(\alpha_1 u^4 + 4\beta_1 u^3 v + 6\gamma_1 u^2 v^2 + 4\delta_1 uv^3 + \epsilon_1 v^4) \\ &\quad + \tfrac{1}{120}(\alpha_1' u^5 + 5\beta_1' u^4 v + 10\gamma_1' u^3 v^2 + 10\delta_1' u^2 v^3 + 5\epsilon_1' uv^4 + \zeta_1' v^5) + \cdots, \\ x_2 &= u - \tfrac{1}{6}fu^3 - \tfrac{1}{2}guv^2 + \tfrac{1}{24}(\alpha_2 u^4 + 4\beta_2 u^3 v + 6\gamma_2 u^2 v^2 + 4\delta_2 uv^3 + \epsilon_2 v^4) \\ &\quad + \tfrac{1}{120}(\alpha_2' u^5 + 5\beta_2' u^4 v + 10\gamma_2' u^3 v^2 + 10\delta_2' u^2 v^3 + 5\epsilon_2' uv^4 + \zeta_2' v^5) + \cdots \\ x_3 &= v - bu^2 - \tfrac{1}{3}b_u u^3 - \tfrac{1}{2}(f + 2b_v)u^2 v - \tfrac{1}{6}gv^3 \\ &\quad + \tfrac{1}{24}(\alpha_3 u^4 + 4\beta_3 u^3 v + 6\gamma_3 u^2 v^2 + 4\delta_3 uv^3 + \epsilon_3 v^4) \end{aligned}$$

$$+ \tfrac{1}{120}(\alpha_3' u^5 + 5\beta_3' u^4 v + 10\gamma_3' u^3 v^2 + 10\delta_3' u^2 v^3 + 5\epsilon_3' u v^4 + \zeta_3' v^5) + \cdots,$$

$$x_4 = uv - \tfrac{1}{3} b u^3 + \tfrac{1}{24}(\alpha_4 u^4 + 4\beta_4 u^3 v + 6\gamma_4 u^2 v^2 + 4\delta_4 u v^3 + \epsilon_4 v^4)$$
$$+ \tfrac{1}{120}(\alpha_4' u^5 + 5\beta_4' u^4 v + 10\gamma_4' u^3 v^2 + 10\delta_4' u^2 v^2 + 5\epsilon_4' u v^4 + \zeta_4' v^5) + \cdots.$$

We find therefore

$$(12)\quad \frac{1}{x_1} = 1 + \tfrac{1}{2} f u^2 + \tfrac{1}{2} g v^2 + \tfrac{1}{6} f_u u^3 - (bg - \tfrac{1}{2} f_v) u^2 v + \tfrac{1}{6} g_v v^3 + \tfrac{1}{24}(6f^2 - \alpha_1) u^4$$
$$- \tfrac{1}{6}\beta_1 u^3 v + \tfrac{1}{4}(2fg - \gamma_1) u^2 v^2 - \tfrac{1}{6}\delta_1 u v^3 + \tfrac{1}{24}(6g^2 - \epsilon_1) v^4 + \cdots;$$

whence, if we introduce non-homogeneous coördinates

$$\xi = \frac{x_2}{x_1} = u + \tfrac{1}{3} f u^3 + \tfrac{1}{24}(4 f_u + \alpha_2) u^4 + \tfrac{1}{6}(\beta_2 - 6bg + 3f_v) u^3 v + \tfrac{1}{4}\gamma_2 u^2 v^2$$
$$+ \tfrac{1}{6}(g_v + \delta_2) u v^3 + \tfrac{1}{24}\epsilon_2 v^4 + \tfrac{1}{120}(20f^2 - 5\alpha_1 + \alpha_2') u^5$$
$$+ \tfrac{1}{24}(\beta_2' - 4\beta_1) u^4 v + \tfrac{1}{12}(2fg + \gamma_2' - 3\gamma_1) u^3 v^2 + \tfrac{1}{12}(\delta_2' - 2\delta_1) u^2 v^3$$
$$+ \tfrac{1}{24}(\epsilon_2' - \epsilon_1) u v^4 + \tfrac{1}{120}\zeta_2' v^5 + \cdots,$$

$$\eta = \frac{x_3}{x_1} = v - bu^2 - \tfrac{1}{3} b_u u^3 - b_v u^2 v + \tfrac{1}{3} g v^3 + \tfrac{1}{24}(\alpha_3 - 12bf) u^4 + \tfrac{1}{6}(f_u + \beta_3) u^3 v$$
$$+ \tfrac{1}{4}(2f_v - 6bg + \gamma_3) u^2 v^2 + \tfrac{1}{6}\delta_3 u v^3 + \tfrac{1}{24}(\epsilon_3 + 4g_v) v^4 + \tfrac{1}{120}(\alpha_3' - 40bf_u) u^5$$
$$(13)\quad + \tfrac{1}{24}(\beta_3' - \alpha_1 + 24b^2 g - 12bf_v - 12b_u f) u^4 v + \tfrac{1}{12}(\gamma_3' - 2\beta_1 - 2b_u g) u^3 v^2$$
$$+ \tfrac{1}{12}(\delta_3' - 3\gamma_1 + 2fg - 2g_v b - 6gb_v) u^2 v^3 + \tfrac{1}{24}(\epsilon_3' - 4\delta_1) u v^4$$
$$+ \tfrac{1}{120}(\zeta_3' - 5\epsilon_1 + 20g^2) v^5 + \cdots,$$

$$\zeta = \frac{x_4}{x_1} = uv - \tfrac{1}{3} b u^3 + \tfrac{1}{24}\alpha_4 u^4 + \tfrac{1}{6}(\beta_4 + 3f) u^3 v + \tfrac{1}{4}\gamma_4 u^2 v^2 + \tfrac{1}{6}(\delta_4 + 3g) u v^3$$
$$+ \tfrac{1}{24}\epsilon_4 v^4 + \tfrac{1}{120}(\alpha_4' - 20bf) u^5 + \tfrac{1}{24}(\beta_4' + 4f_u) u^4 v$$
$$+ \tfrac{1}{12}(\gamma_4' - 14bg + 6f_v) u^3 v^2 + \tfrac{1}{12}\delta_4' u^2 v^3 + \tfrac{1}{24}(\epsilon_4' + 4g_v) u v^4$$
$$+ \tfrac{1}{120}\zeta_4' v^5 + \cdots.$$

This gives

$$\zeta - \xi\eta = \tfrac{2}{3} b u^3 + \tfrac{1}{6} b_u u^4 + \tfrac{2}{3} b_v u^3 v + \tfrac{1}{30}(b_{uu} + 16bf - 4bb_v) u^5 + \tfrac{1}{6} b_{uv} u^4 v$$
$$+ \tfrac{1}{6}(4bg - 3f_v - b_{vv}) u^3 v^2 + \cdots,$$

$$(14)\quad \xi^3 = u^3 + f u^5 + \cdots, \qquad \xi^3\eta = u^3 v - b u^5 + \cdots,$$
$$\xi^4 = u^4 + \cdots, \qquad \xi^3\eta^2 = u^3 v^2 + \cdots,$$
$$\xi^5 = u^5 + \cdots, \qquad \xi^4\eta = u^4 v + \cdots,$$

so that we obtain the following development

$$\zeta = \xi\eta + \tfrac{2}{3}b\xi^3 + \tfrac{1}{6}b_u\xi^4 + \tfrac{2}{3}b_v\xi^3\eta + \tfrac{1}{30}(b_{uu} - 4bf + 16bb_v)\xi^5 + \tfrac{1}{6}b_{uv}\xi^4\eta + \tfrac{1}{6}(4bg - 3f_v - b_{vv})\xi^3\eta^2 + \cdots. \tag{15}$$

Notice that, in this series, the terms which involve ξ^0, ξ, η, ξ^2, η^2, $\xi^2\eta$, $\xi\eta^2$, η^3, $\xi^2\eta^2$, $\xi\eta^3$, η^4, $\xi^2\eta^3$, $\xi\eta^4$, η^4 are absent, while the coefficient of $\xi\eta$ is unity. Just as in the case of a non-ruled surface, it may be shown that the most general transformation of coördinates which preserves this form of the development is the following:

$$x = \frac{\lambda(\xi + \beta\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad y = \frac{\mu(\eta + \alpha\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad z = \frac{\lambda\mu\zeta}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \tag{16}$$

where α, β, λ, μ are constants at our disposal, which we shall choose in such a way as to simplify further the form of the development.

We find, to terms of the third order inclusive,

$$\frac{1}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta} = 1 - \alpha\xi - \beta\eta + \alpha^2\xi^2 + \alpha\beta\xi\eta + \beta^2\eta^2 - (\alpha^3 + \tfrac{2}{3}b\alpha\beta)\xi^3 - \alpha^2\beta\xi^2\eta - \alpha\beta^2\xi\eta^2 - \beta^3\eta^3 + \cdots, \tag{17}$$

so that we shall have, to terms of the fourth order,

$$\begin{aligned} \frac{x}{\lambda} &= \xi - \alpha\xi^2 + (\alpha^2 + \tfrac{2}{3}b\beta)\xi^3 + \tfrac{1}{6}(-8b\alpha\beta - 6\alpha^3 + b_u\beta)\xi^4 + \tfrac{2}{3}(-b\beta^2 + b_v\beta)\xi^3\eta + \cdots, \\ \frac{y}{\mu} &= \eta - \beta\eta^2 + \tfrac{2}{3}\alpha b\xi^3 + \beta^2\eta^3 + \tfrac{1}{6}(-4\alpha^2 b + b_u\alpha)\xi^4 + \tfrac{2}{3}(-2b\alpha\beta + b_v\alpha)\xi^3\eta - \beta^3\eta^4 + \cdots, \end{aligned} \tag{18}$$

whence, to terms of the fifth order inclusive,

$$\begin{aligned} \frac{xy}{\lambda\mu} &= \xi\eta - \alpha\xi^2\eta - \beta\xi\eta^2 + \tfrac{2}{3}\alpha b\xi^4 + (\alpha^2 + \tfrac{2}{3}b\beta)\xi^3\eta + \alpha\beta\xi^2\eta^2 + \beta^2\xi\eta^3 \\ &\quad + \tfrac{1}{6}\alpha(b_u - 8b\alpha)\xi^5 + \tfrac{1}{6}(b_u\beta + 4b_v\alpha - 16b\alpha\beta - 6\alpha^3)\xi^4\eta \\ &\quad + \tfrac{1}{3}(-4b\beta^2 + 2b_v\beta - 3\alpha^2\beta)\xi^3\eta^2 - \alpha\beta^2\xi^2\eta^3 - \beta^3\xi\eta^4 + \cdots, \\ \frac{z}{\lambda\mu} &= \xi\eta + \tfrac{2}{3}b\xi^3 - \alpha\xi^2\eta - \beta\xi\eta^2 + \tfrac{1}{6}(b_u - 4\alpha b)\xi^4 + \tfrac{1}{3}(3\alpha^2 - 2b\beta + 2b_v)\xi^3\eta \\ &\quad + \alpha\beta\xi^2\eta^2 + \beta^2\xi\eta^3 + \tfrac{1}{30}(b_{uu} - 4bf + 16bb_v - 5b_u\alpha + 20b\alpha^2)\xi^5 \\ &\quad + \tfrac{1}{6}(-6\alpha^3 - b_u\beta - 4b_v\alpha + b_{uv})\xi^4\eta + \tfrac{1}{6}(4bg - 3f_v - b_{vv} - 6\alpha^2\beta \\ &\quad + 4b\beta^2 - 4b_v\beta)\xi^3\eta^2 - \alpha\beta^2\xi^2\eta^3 - \beta^3\xi\eta^4 + \cdots. \end{aligned} \tag{19}$$

Consequently we obtain:

$$\begin{aligned}\frac{z-xy}{\lambda\mu} = {} & \tfrac{2}{3}b\xi^3 + \tfrac{1}{6}(b_u - 8ab)\xi^4 + \tfrac{2}{3}(b_v - 2b\beta)\xi^3\eta + \tfrac{1}{3\,6}(b_{uu} - 4bf \\ & + 16bb_v - 10b_u\alpha + 60b\alpha^2)\xi^5 + \tfrac{1}{6}(b_{uv} - 8b_v\alpha - 2b_u\beta + 16b\alpha\beta)\xi^4\eta \\ & + \tfrac{1}{6}(4bg - 3f_v - b_{vv} - 8b_v\beta + 12b\beta^2)\xi^3\eta^2 + \cdots.\end{aligned} \tag{20}$$

Again we find, to terms of the fifth order inclusive,

$$\left(\frac{x}{\lambda}\right)^3 = \xi^3 - 3\alpha\xi^4 + (6\alpha^2 + 2b\beta)\xi^5 + \cdots,$$

$$\left(\frac{x}{\lambda}\right)^4 = \xi^4 - 4\alpha\xi^5 + \cdots, \quad \left(\frac{x}{\lambda}\right)^5 = \xi^5 + \cdots, \quad \left(\frac{x}{\lambda}\right)^4\frac{y}{\mu} = \xi^4\eta + \cdots,$$

$$\left(\frac{x}{\lambda}\right)^3\left(\frac{y}{u}\right)^2 = \xi^3\eta^2\cdots, \quad \left(\frac{x}{\lambda}\right)^2\left(\frac{y}{\mu}\right)^3 = \xi^2\eta^3 + \cdots, \quad \frac{x}{\lambda}\left(\frac{y}{\mu}\right)^4 = \xi\eta^4 + \cdots$$

whence

$$\xi^4 = \left(\frac{x}{\lambda}\right)^4 + 4\alpha\left(\frac{x}{\lambda}\right)^5 + \cdots,$$

$$\xi^3 = \left(\frac{x}{\lambda}\right)^3 + 3\alpha\left(\frac{x}{\lambda}\right)^4 + (6\alpha^2 - b_v)\left(\frac{x}{\lambda}\right)^5 + \cdots.$$

Let us substitute these values in (20), and put

$$\beta = \frac{b_v}{2b} \tag{21}$$

as in the case of a non-ruled surface. Then (20) becomes

$$\begin{aligned}\frac{z-xy}{\lambda\mu} = {} & \tfrac{2}{3}b\left(\frac{x}{\lambda}\right)^3 + \tfrac{1}{6}(b_u + 4b\alpha)\left(\frac{x}{\lambda}\right)^4 + \tfrac{1}{3\,6}(20b\alpha^2 + 10b_u\alpha + b_{uu} - 4bf - 4bb_v)\left(\frac{x}{\lambda}\right)^5 \\ & + \tfrac{1}{6}\left(b_{uv} - \frac{b_u b_v}{b}\right)\left(\frac{x}{\lambda}\right)^4\frac{y}{\mu} + \tfrac{1}{6}\left(4bg - 3f_v - b_{vv} - \frac{b_v^2}{b}\right)\left(\frac{x}{\lambda}\right)^3\left(\frac{y}{\mu}\right)^2 + \cdots.\end{aligned} \tag{22}$$

We may choose α so as to make the coefficient of $(x/\lambda)^4$ equal to zero, i. e., we may put

$$\alpha = -\frac{b_u}{4b}. \tag{23}$$

Then the series becomes

$$\frac{z-xy}{\lambda\mu} = \tfrac{2}{3}b\left(\frac{x}{\lambda}\right)^3 - \tfrac{2}{1\,5}\frac{h}{b}\left(\frac{x}{\lambda}\right)^5 + \frac{C'}{48}\left(\frac{x}{\lambda}\right)^4\frac{y}{\mu} - \frac{\theta'}{384b}\left(\frac{x}{\lambda}\right)^3\left(\frac{y}{\mu}\right)^2 + \cdots, \tag{24}$$

where h, C', and θ' are the same quantities (invariants) which have been denoted by these symbols in the first and second memoirs, viz.;

$$h = b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2,$$

$$\theta' = 64\,[b_v^2 + 2bf_v - 4b^2 g\,], \tag{25}$$

$$C' = 8\left(b_{uv} - \frac{b_u b_v}{b}\right).$$

Of course, in making the substitutions (21) and (23) it is assumed that b is not equal to zero; this means that the point of the ruled surface, in the vicinity of which the development (24) is valid, is not a flecnode.* Any ruled surface which is not a quadric will therefore admit of a development of the form (24) in the vicinity of any one of its ordinary points which is not a flecnode, since the quadric is the only ruled surface all of whose points are flecnodes.

The constants λ and μ are still at our disposal. We shall choose them in such a way as to make the coefficients of the development absolute invariants. If h is different from zero, assume

$$\lambda^2 = \tfrac{2}{3}b\mu, \qquad \lambda^4 = -\frac{2h}{15b}\mu,$$

whence

$$\lambda = \frac{1}{b}\sqrt{-\tfrac{1}{5}h}, \qquad u = -\frac{3h}{10b^3}, \tag{26}$$

where either of its two values may be given to the square root. This gives us the final form of the canonical development for $h \neq 0$,

$$z = xy + x^3 + x^5 - \frac{5b^3C'}{48h\sqrt{-\frac{1}{5}h}}x^4y - \frac{5^2b^4\theta'}{2^6\cdot 3^2h^2}x^3y^2 + \cdots. \tag{27}$$

The equation of any ruled surface may be developed in two ways into a power-series of form (27) *in the vicinity of any ordinary point of the surface which is not a flecnode and for which the invariant h does not vanish.*

If $h = 0$ while $b \neq 0$, (24) is still valid. Assume $C' \neq 0$, and determine λ and μ so that

$$\lambda^2 = \tfrac{2}{3}b\mu, \qquad \lambda^3 = \frac{C'}{48},$$

whence

$$\lambda = \tfrac{1}{2}\sqrt[3]{\tfrac{1}{b}C'}, \qquad \mu = \frac{3\sqrt[3]{\frac{1}{36}C'^2}}{8b},$$

which gives *the canonical development for this case* ($h = 0$, $C' \neq 0$),

*Cf. second memoir, eq. (15). This gives $u'_{12} = -8b$. But $u'_{12} = 0$ is the condition that P_v be a flecnode. (*Proj. Diff. Geom.*, p. 150.)

$$z = xy + x^3 + x^4 y - \frac{\theta'}{6C'\sqrt[4]{\frac{1}{6}C'}} x^3 y^2 + \cdots. \tag{28}$$

If $h = C' = 0$, $\theta' \neq 0$ we put

$$\lambda^2 = \tfrac{2}{3} b\mu, \qquad \lambda^2 \mu = -\frac{\theta'}{384 b},$$

with the resulting development

$$z = xy + x^3 + x^3 y^2 + \cdots. \tag{29}$$

If $h = C' = \theta' = 0$ the establishment of a definite canonical form would require the computation of terms of the sixth order. We shall see, however, in a later paragraph that if the conditions

$$b \neq 0, \qquad h = C' = \theta' = 0,$$

are fulfilled identically, not only will the development

$$z = xy + x^3 \tag{30}$$

be exact up to terms of the fifth order, inclusive, but the series will break off at this point, so that the surface is of the third degree. Finally if the condition $b = 0$ is fulfilled identically, the surface is a quadric

$$z = xy. \tag{31}$$

Every ruled surface may be represented by at least one of the developments (27), (28), (29), (30), *or* (31) *in the vicinity of one of its ordinary points. The conditions under which one or the other of these developments breaks down are given by the vanishing of one or several of the invariants* b, h, c', θ'. *The geometrical significance of these conditions will become apparent subsequently.*

§ 2. *Geometrical determination of the canonical tetrahedron. The osculating Cayley's cubic scroll.*

The canonical development having been found, we proceed to investigate the geometrical significance of the tetrahedron of reference which is connected with it. As in the case of the non-ruled surfaces it becomes necessary, for this purpose, to consider a unodal cubic surface which has contact of the fourth order with the given surface S at the given point P.

The equation

$$Z(\omega + pX + qY + rZ)^2 + \phi(X + \lambda Z,\ Y + \mu Z,\ \omega + pX + qY + rZ) = 0, \tag{32}$$

in which p, q, r, λ, μ are arbitrary constants and where ϕ denotes a general ternary cubic involving ten further arbitrary constants, represents the most general cubic surface with a unode. The intersection of the three planes

$$x + \lambda Z = 0, \qquad Y + \mu Z = 0, \qquad \omega + pX + qY + rZ = 0 \tag{33}$$

will be its unode, and the plane

$$\omega + pX + qY + rZ = 0 \tag{34}$$

its uniplane.*

We may introduce homogeneous coördinates into (27) by putting

$$x = \frac{X}{\omega}, \qquad y = \frac{Y}{\omega}, \qquad z = \frac{Z}{\omega}. \tag{35}$$

We intend to determine the fifteen constants of (32) in such a way that the unodal cubic shall have contact of the fourth order with S at P. Then, to begin with, the plane $Z = 0$ (the tangent plane) must intersect the cubic surface in a curve which has a double point at $X = Y = 0$ (the point of contact), the tangents at the double point being $X = 0$ and $Y = 0$ respectively (the two asymptotic tangents of the surface S at the point P). Now the curve, in which $Z = 0$ intersects the cubic surface (32), is

$$Z = 0, \qquad \phi(X, Y, \omega + pX + qY) = 0.$$

The conditions just mentioned require that ϕ shall be of the form

$$aX^3 + bY^3 + dX^2Y + eXY^2 + lXY(\omega + pX + qY),$$

so that our unodal cubic will have an equation of the form

$$\begin{aligned} &Z(\omega+pX+qY+rZ)^2+a(X+\lambda Z)^3+b(Y+\mu Z)^3+d(X+\lambda Z)^2(Y+\mu Z) \\ &\quad +e(X+\lambda Z)(Y+\mu Z)^2+l(X+\lambda Z)(Y+\mu Z)(\omega+pX+qY+rZ)=0, \end{aligned}$$

or in non-homogeneous coördinates

$$\begin{aligned} &z(1+px+qy+rz)^2+a(x+\lambda z)^3+b(y+\mu z)^3+d(x+\lambda z)^2(y+\mu z) \\ &\quad +e(x+\lambda z)(y+\mu z)^2+l(x+\lambda z)(y+\mu z)(1+px+qy+rz)=0. \end{aligned} \tag{36}$$

If this cubic surface has fourth order contact with S at P the development (27), when substituted for z in (36), must satisfy the equation identically up to terms of the fourth order inclusive in x and y. The conditions that this be so are found to be as follows:

$$\begin{gathered} l+1=0, \qquad a+1=0, \qquad b=0, \\ 2p+d+l\mu+pl=0, \qquad 2q+e+l(\lambda+q)=0, \\ p^2+2q+3a\lambda+d\mu+l(\lambda+p\mu)=0, \\ 2(pq+r)+2\lambda d+2\mu e+l(\lambda\mu+p\lambda+q\mu+r)=0, \\ q^2+3b\mu+e\lambda+lq\lambda=0, \qquad 2p+l\mu=0, \end{gathered} \tag{37}$$

*Second Memoir, eq. (104).

the solution of which offers no difficulty. We find:

$$(38)\quad a=-1,\ b=0,\ l=-1,\ \lambda=\tfrac{1}{2}p^2,\ \mu=2p,\ d=p,\ e=0,\ q=\tfrac{1}{2}p^2,\ r=\tfrac{1}{2}p^3,$$

so that (in homogeneous form),

$$(39)\quad \begin{aligned} F\equiv\ & Z(\omega+pX+\tfrac{1}{2}p^2Y+\tfrac{1}{2}p^3Z)^2-(X+\tfrac{1}{2}p^2Z)^3+p(X+\tfrac{1}{2}p^2Z)^2(Y+2pZ)\\ & -(X+\tfrac{1}{2}p^2Z)(Y+2pZ)(\omega+pX+\tfrac{1}{2}p^2Y+\tfrac{1}{2}p^3Z)=0 \end{aligned}$$

is the equation of the most general unodal cubic surface which has contact of the fourth order with the ruled surface S at one of its points P. Since p remains arbitrary, we see that there exists a singly infinite family of such surfaces.

In order to investigate these surfaces a little more in detail, let us put

$$(40)\quad \xi=X+\tfrac{1}{2}p^2Z,\quad \eta=Y+2pZ,\quad \zeta=Z,\quad \Omega=\omega+pX+\tfrac{1}{2}p^2Y+\tfrac{1}{2}p^3Z,$$

so that (39) becomes

$$(41)\qquad F\equiv\zeta\Omega^2-\xi^3+p\xi^2\eta-\xi\eta\Omega=0.$$

Indicate the partial derivatives of F by subscripts. Then

$$F_\xi=-3\xi^2+2p\xi\eta-\eta\Omega,\qquad F_\eta=p\xi^2-\xi\Omega,$$
$$F_\zeta=\Omega^2,\qquad F_\Omega=2\zeta\Omega-\xi\eta.$$

These equations show that (41) has a nodal line $\xi=\Omega=0$ for every value of p, so that all of these cubic surfaces are scrolls. The locus of these nodal lines

$$\xi=X+\tfrac{1}{2}p^2Z=0,\qquad \Omega=\omega+\tfrac{1}{2}p^2Y+p(X+\tfrac{1}{2}p^2Z)=0,$$

obtained by eliminating p, is the hyperboloid

$$\omega Z-XY=0$$

which osculates the ruled surface S along the generator g that passes through P. We find further

$$F_{\xi\xi}=-6\xi+2p\eta,\qquad F_{\eta\eta}=0,\qquad F_{\zeta\zeta}=0,\qquad F_{\Omega\Omega}=2\zeta.$$
$$F_{\xi\eta}=2p\xi-\Omega,\qquad F_{\eta\zeta}=0,\qquad F_{\zeta\Omega}=2\Omega,$$
$$F_{\xi\zeta}=0,\qquad F_{\eta\Omega}=-\xi,$$
$$F_{\xi\Omega}=-\eta.$$

Let $(0,\eta',\zeta',0)$ be a point of the nodal line. The equation of the pair of planes tangent to (41) at this point will be

$$\eta'(p\xi^2 - \xi\Omega) + \zeta'\Omega^2 = 0, \tag{42}$$

which will coincide if and only if

$$\eta'(\eta' - 4p\zeta') = 0, \tag{43}$$

so that the ruled cubic will have two distinct unodes

$$(0, 0, 1, 0) \quad \text{and} \quad (0, 4p, 1, 0),$$

the two *pinch points* of the cubic scroll, if p is not equal to zero. If p is equal to zero, the two pinch-points coincide, so that we obtain a CAYLEY's *cubic scroll*. Equation (42) then becomes

$$\Omega(-\eta'\xi + \zeta'\Omega) = 0$$

or

$$\omega(-Y'X + Z'\omega) = 0,$$

so that the plane $\omega = 0$ is tangent to the ruled cubic along the whole nodal line, while the other plane touches it at just one point. We shall speak of $\omega = 0$ as the *singular tangent plane* of the CAYLEY cubic. The equation of the CAYLEY cubic itself becomes

$$Z\omega^2 - X^3 - XY\omega = 0, \tag{44}$$

or in non-homogeneous coördinates

$$z = xy + x^3. \tag{45}$$

If we compare this with the canonical development (27), it becomes apparent that one of the faces of the canonical tetrahedron is the singular tangent plane, and one of its vertices the pinch-point of the osculating CAYLEY cubic. This gives rise to the following theorem.

1. *The unodal cubic surfaces which have fourth order contact with a given ruled surface S at a given point P, which is not a flecnode, are themselves ruled surfaces. They form a one-parameter family. Just one surface of this family is a Cayley cubic scroll, and may be called the osculating Cayley cubic scroll of the point P of the ruled surface S.*

2. *The canonical tetrahedron has the given point P of the ruled surface S as one vertex, and the two asymptotic tangents of that point (one of which is the generator g of S), as two of its edges. The pinch-point of the osculating Cayley cubic scroll, its nodal line, and its singular tangent plane constitute three further elements of the canonical tetrahedron, which is thus completely determined geometrically. The nodal line of the osculating Cayley cubic scroll, moreover, meets the generator g of S.*

§ 3. *The family of cubic scrolls having fourth order contact with the ruled surface S at the point P.*

We have noted already that the locus of the nodal lines of the cubic scrolls (41) is the osculating hyperboloid. What is the locus of their pinch-points? We have seen that the pinch-points are determined by the conditions

$$\xi = \Omega = 0, \qquad \eta(\eta - 4p\zeta) = 0,$$

where ξ, η, ζ, Ω are defined by (40). We find, therefore, that the coördinates of these two pinch-points are given by

$$(46) \qquad X_1 = -\tfrac{1}{2}p^2, \qquad Y_1 = -2p, \qquad Z_1 = 1, \qquad \omega_1 = p^3,$$

and

$$(47) \qquad X_2 = -\tfrac{1}{2}p^2, \qquad Y_2 = +2p, \qquad Z_2 = 1, \qquad \omega_2 = -p^3,$$

respectively. For $p = 0$, of course, the two points coincide. It is easy to verify that both $X_1, \cdots, \omega_1$ and $X_2, \cdots, \omega_2$ satisfy the three quadratic equations

$$(48) \qquad \omega Z - XY = 0, \qquad Y^2 + 8XZ = 0, \qquad 8X^2 + Y\omega = 0,$$

so that the locus of the two pinch-points is a single twisted cubic curve upon the osculating hyperboloid. Let us speak of that set of generators of the osculating hyperboloid to which the generator g of the given ruled surface belongs as being of the *first kind*. It will then be apparent that the following theorem is true.

The nodal lines of the cubic scrolls, which have fourth order contact with a given ruled surface S at a given point P, are the generators of the second kind upon the hyperboloid H, which osculates S along the generator g which passes through P. The pinch-points of these cubic scrolls are the points of a twisted cubic curve on H which intersects every generator of the second kind in two points. This twisted cubic passes through P, is tangent at P to the curved asymptotic line of S which passes through that point, and has the plane tangent to S at P as its osculating plane. It also passes through the pinch-point of the osculating Cayley cubic scroll and there has the nodal line and singular tangent plane of that surface as its tangent and osculating plane respectively.

We may notice further that the two cubic surfaces of the one-parameter family, for which the parameter p has numerically equal but opposite values, have the same nodal lines and the same pair of pinch-points, as follows from (46) and (47). This may be expressed as follows:

There exists a two to two correspondence between the cubic surfaces of our one-parameter family and the points of the twisted cubic.

Finally it may be verified that the *locus of the two planes, which are tangent to one of the cubic scrolls of the one-parameter family at its pinch-points, is a developable whose cuspidal edge is again a twisted cubic curve.*

§ 4. *Osculating pseudo-paraboloids of various orders.*

If x and y are cartesian coördinates, a curve whose equation is of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is sometimes known as a parabola of the nth order. Similarly we might speak of a surface whose cartesian equation is of the form

$$z = a_0 + a_1 x + a_2 y + \cdots + u_n,$$

where u_n is homogeneous and of the nth degree in x and y, as a paraboloid of the nth order. This latter equation may be written in the homogeneous form

$$Z\omega^{n-1} + u_0\omega^n + u_1\omega^{n-1} + u_2\omega^{n-2} + \cdots + u_{n-1}\omega + u_n = 0,$$

where u_k is a binary form of degree k in X and Y, and where $\omega = 0$ represents the plane at infinity. We shall have to consider surfaces of the nth order whose equations are of this form but in which the plane $\omega = 0$ is not the plane at infinity and in which the three planes $X = 0$, $Y = 0$, $Z = 0$ are not mutually perpendicular. These surfaces, the projective generalizations of the paraboloids of the nth order, shall be known as pseudo-paraboloids. We may write the equation of the surface in the form

$$Z\omega^{n-1} + \phi(X, Y, \omega) = 0,$$

where $\phi(X, Y, \omega)$ is the most general ternary form of the nth order in the variables X, Y, ω, and in which Z does not occur. *The point* $X = Y = \omega = 0$, $Z = 1$ *is an* $n - 1$ *fold point of the surface whose tangent cone degenerates into the plane* $\omega = 0$ *counted* $n - 1$ *times. This property is characteristic of pseudo-paraboloids.* We may speak of the singular point as a *unode of the* $n - 1$*th order*, and of the plane as its *uniplane.*

If the canonical development (27) be broken off with the terms of any finite degree n, a pseudo-paraboloid of the nth order will be obtained, whose unode and uniplane coincide with those of the osculating Cayley cubic, which has contact of the nth order with the given ruled surface at the point considered, and which is uniquely determined by these conditions.

We shall make use of one of these pseudo-paraboloids in the following consideration. We have already ascertained the geometrical significance of the tetrahedron which gives rise to the canonical development. But this does not suffice to determine the coördinates themselves. In fact any transformation of the form

$$X = \lambda_1 \overline{X},\ Y = \lambda_2 \overline{Y},\ Z = \lambda_3 \overline{Z},\ \omega = \lambda_4 \overline{\omega},$$

or, in non-homogeneous coördinates,

$$x = \lambda \bar{x},\ y = \mu \bar{y},\ z = \nu \bar{z}, \tag{49}$$

where λ, μ, ν are constants will leave unaltered the tetrahedron of reference. However, in the system of coördinates as employed by us, the equations of the osculating hyperboloid and of the osculating Cayley cubic were

$$z - xy = 0 \qquad \text{and} \qquad z - xy - x^3 = 0$$

respectively. These equations are left invariant by (49) only if

$$\nu = \lambda\mu, \qquad \lambda^2 = \mu,$$

i. e., if

$$\mu = \lambda^2, \qquad \nu = \lambda^3. \tag{50}$$

But the osculating pseudo-paraboloid of the fifth order was also endowed with an absolutely fixed equation, and we find that λ must be equated to unity in (49) and (50) so that this equation also may be left invariant. Therefore, *the constants, which are still available after the tetrahedron of the canonical system of coördinates has been selected, are chosen in such a way that the osculating hyperboloid, the osculating Cayley cubic scroll, and the osculating pseudo-paraboloid of the fifth order may have the equations*

$$z - xy = 0, \qquad z - xy - x^3 = 0,$$

$$z - xy - x^3 - x^5 + \frac{5b^3C'}{48h\sqrt{-\frac{1}{5}h}}x^4y + \frac{5^2b^4\theta'}{2^6\cdot 3^2h^2}x^3y^2 = 0, \tag{51}$$

respectively.

§ 5. *Tangents and curves of fifth-order contact. Discussion of the exceptional cases.*

The osculating Cayley cubic scroll has contact of the fourth order with S at the point P, i. e., all sections of the two surfaces made by a plane through P will have contact of at least the fourth order. It is apparent, however, from (27) that there will be certain tangents of S at P such that, in their directions, the contact of the two surfaces will be of the fifth order at least. We shall speak of these as the tangents of fifth-order contact. The equation of these tangents, obtained by equating to zero the terms of the fifth order in (27), shows that there are five such tangents, of which, however, the generator g of S counts as three. The other two are given by the equation

$$x^2 - \frac{5b^3C'}{48h\sqrt{-\frac{1}{5}h}}xy - \frac{5^2b^4\theta'}{2^6\cdot 3^2\cdot h^2}y^2 = 0, \tag{52}$$

and may be spoken of more specifically as the *proper tangents of fifth order contact.* They obviously determine two families of ∞^1 curves on the surface, two of which (in general) pass through every point of the surface. They may be called the *curves of fifth-order contact.*

A few special cases are of interest. *If* $h = 0$, *while* b, C', θ' *do not vanish, one of the tangents of fifth-order contact coincides with the asymptotic tangent of the point* P [cf., eq. (24)]. *If* $C' = 0$, *while* b, h, *and* θ' *are non-vanishing quantities, the two proper tangents of fifth order contact separate the generator and the asymptotic tangent harmonically. If* $\theta' = 0$ *while* b, h, C' *are different from zero, one of the proper tangents of fifth-order contact coincides with the generator, which in that case counts for four. This latter case is to be well distinguished, of course, from that in which the two proper tangents of fifth-order contact coincide, i. e., in which the discriminant of* (52) *is equal to zero.* Let us remember that the condition $\theta' = 0$ characterizes those generators of the ruled surface whose flecnodes coincide. Therefore: *if the two flecnodes of a generator coincide, four of the five tangents of fifth-order contact at every point of that generator coincide with the generator itself, and conversely.*

If $b \neq 0$, $\theta' \neq 0$, $h = C' = 0$, *two tangents of fifth-order contact coincide with the asymptotic tangent, the other three, of course, with the generator. Finally if*

$$h = C' = \theta' = 0, \qquad b \neq 0, \tag{53}$$

the fifth-order contact tangents become indeterminate, i. e., in this case the Cayley cubic scroll has contact of at least the fifth order with the given ruled surface. If the conditions (53) *are satisfied, not merely at a particular point* u, v *of the surface but identically, the surface must itself be a Cayley cubic scroll.*

It is of interest to prove this directly. If the conditions (53) are satisfied identically, we shall have, making use of the expressions (25) for h, C' and θ',

$$b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2 = 0, \qquad b_v^2 + 2bf_v - 4b^2 g = 0,$$
$$b_{uv} - \frac{b_u b_v}{b} = 0. \tag{54}$$

If $b_u \neq 0$, the last equation may be written

$$\frac{b_{uv}}{b_u} = \frac{b_v}{b},$$

whence

$$b = \phi(u)\psi(v), \tag{55}$$

where $\phi(u)$ and $\psi(v)$ are functions of u and v alone respectively. If $b_u = 0$, b is a function of v alone, which case is also included in (55) if there we put $\phi(u) = 1$.

The substitution of (55) into the first equation of (54) gives

$$f = \frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2} - \phi\psi_v. \tag{56}$$

On account of the integrability conditions (2), g is a function of v alone, say

$$g = \chi(v),$$

so that the second equation (54) gives

$$\phi^2\psi_v^2 - 2\phi^2\psi\psi_{vv} - 4\phi^2\psi^2\chi = 0.$$

Since $b \neq 0$, neither ϕ nor ψ can be identically zero, so that

$$\psi_v^2 - 2\psi\psi_{vv} - 4\psi^2\chi = 0. \tag{57}$$

The two remaining integrability conditions (2), or

$$b_{vv} + f_v = 0, \qquad -f_{vv} + 4gb_v + 2bg_v = 0,$$

are satisfied. This is obvious for the first. The second gives, after division by ϕ,

$$\psi_{vvv} + 4\chi\psi_v + 2\psi\chi_v = 0,$$

which is satisfied because this same equation can be obtained from (57) by differentiation.

The differential equations of the surface become, therefore,

$$\begin{gathered} y_{uu} + 2\phi(u)\psi(v)y_v + \left[\frac{1}{4}\frac{\phi_{uu}}{\phi} - \frac{5}{16}\frac{\phi_u^2}{\phi^2} - \phi\psi_v\right]y = 0, \\ y_{vv} - \frac{1}{2}\left(\frac{\psi_{vv}}{\psi} - \frac{1}{2}\frac{\psi_v^2}{\psi^2}\right)y = 0; \end{gathered} \tag{58}$$

this system we shall now proceed to integrate.

Let us substitute

$$y = \sqrt{\psi(v)}\,\bar{y}$$

in the second equation of (58). We shall find

$$\bar{y}_{vv} + \frac{\psi_v}{\psi}\bar{y}_v = 0,$$

whence

$$\bar{y} = U + U'\bar{v}, \tag{59}$$

where

$$\bar{v} = \int \frac{dv}{\psi(v)}, \tag{60}$$

and where U and U' are functions of u only. Let us substitute the resulting value of y, i. e.,

$$y = \sqrt{\psi}(U + U'\bar{v}) \tag{61}$$

in the first equation (58). We shall find

$$\sqrt{\psi}(U_{uu}+U'_{uu}\bar{v})+2\phi\sqrt{\psi}\,[U'+\tfrac{1}{2}(U+U'\bar{v})\psi_v]$$
$$+\left(\frac{1}{4}\frac{\phi_{uu}}{\phi}-\frac{5}{16}\frac{\phi_u^2}{\phi^2}-\phi\psi_v\right)\sqrt{\psi}(U+U'\bar{v})=0,$$

or

$$U_{uu}+U'_{uu}\bar{v}+2\phi U'+\left(\frac{1}{4}\frac{\phi_{uu}}{\phi}-\frac{5}{16}\frac{\phi_u^2}{\phi^2}\right)(U+U'\bar{v})=0.$$

Since u and v are independent variables, that part of the left member of this equation which is independent of v and that part which contains $\bar{v}$ as a factor must vanish independently, i. e., we must have

$$U_{uu}+2\phi(u)U'+\left(\frac{1}{4}\frac{\phi_{uu}}{\phi}-\frac{5}{16}\frac{\phi_u^2}{\phi^2}\right)U=0,$$

(62)

$$U'_{uu}+\left(\frac{1}{4}\frac{\phi_{uu}}{\phi}-\frac{5}{16}\frac{\phi_u^2}{\phi^2}\right)U'=0.$$

Put

$$U=\frac{\overline{U}}{\sqrt[4]{\phi(u)}},\qquad U'=\frac{\overline{U}'}{\sqrt[4]{\phi(u)}}.$$

The resulting differential equations for $\overline{U}$ and $\overline{U}'$ are

$$\overline{U}_{uu}-\frac{1}{2}\frac{\phi_u}{\phi}\overline{U}_u+2\phi\overline{U}'=0,$$

$$\overline{U}'_{uu}-\frac{1}{2}\frac{\phi_u}{\phi}\overline{U}'_u=0,$$

whence, if we put

$$\bar{u}=\int\sqrt{\phi(u)}\,du,\tag{63}$$

we obtain

$$\overline{U}'=a\bar{u}+b,$$

$$\overline{U}=-\tfrac{1}{3}a\bar{u}^3-b\bar{u}^2+c\bar{u}+d,$$

where a, b, c, d are arbitrary constants.

The general integral of the system (58) becomes, therefore,

$$y=\frac{\sqrt{\psi(v)}}{\sqrt[4]{\phi(u)}}\,[-\tfrac{1}{3}a\bar{u}^3-b\bar{u}^2+c\bar{u}+d+(a\bar{u}+b)\bar{v}]\tag{64}$$

where

$$\bar{u}=\int\sqrt{\phi(u)}\,du,\qquad \bar{v}=\int\frac{dv}{\psi(v)}.\tag{65}$$

Let us denote the four linearly independent solutions which are multiplied by a, b, c, d respectively by y_1, y_2, y_3, y_4, and let ω be the factor $\phi^{\frac{1}{2}}\psi^{-\frac{1}{2}}$; then

$$(66)\quad \omega y_1 = -\tfrac{1}{3}\bar{u}^3 + \bar{u}\bar{v}, \qquad \omega y_2 = -\bar{u}^2 + \bar{v}, \qquad \omega y_3 = \bar{u}, \qquad \omega y_4 = 1.$$

The elimination of $\bar{u}$ and $\bar{v}$ gives

$$(67)\qquad 3(y_2y_3 - y_1y_4)y_4 - 2y_3^3 = 0,$$

so that the integral surface of (58) is indeed a Cayley cubic scroll as we had intended to show. Incidentally equations (66) show that the asymptotic curves of the Cayley cubic scroll are twisted cubic curves, and that all of them pass through the pinch-point, a known result.

Analytically the following remark is of interest. If $\phi(u)$ and $\psi(v)$ are rational functions of their arguments, the general solution of (58) can always be expressed by hyperelliptic integrals. Let us look upon the variables u and v as being capable of all complex values and construct an appropriate Riemann's surface for each of the integrals (65). The totality of values which a system of solutions of (58) may assume for given values of u and v can be obtained from one of them by such quaternary linear transformations as leave the surface (67) unchanged.

Similar remarks may be made about the general case. Such systems of multiform functions of two independent variables connected with ternary instead of quaternary linear substitutions have been considered to some extent by HORN, PICARD and APPELL.*

§ 5. *The ∞^1 osculating Cayley cubic scrolls which belong to the ∞^1 points of a generator.*

There is associated with every point P of a ruled surface S its osculating Cayley cubic scroll. Let P move along a generator g of S; then the pinch-point of the osculating Cayley cubic will describe a curve, its nodal line will describe a ruled surface, and its singular tangent plane will envelop a developable. We wish to investigate these loci.

In the canonical system of coördinates (X, Y, Z, ω), the coördinates of the pinch-point are $(0, 0, 1, 0)$. Consequently the coördinates of this point referred to our original tetrahedron $P_yP_zP_\rho P_\sigma$ may be easily found. This gives the expression

$$(68)\qquad \pi = -b_ub_vy - 4bb_vz + 2bb_u\rho + 8b^2\sigma$$

for the pinch-point where, be it remembered, y is a solution of the system of differential equations (1) and where z, ρ, σ are defined by (3).

*HORN, Mathematische Annalen, vols. 33 and 34 (1889); Acta Mathematica, vol. 12 (1888-89). APPELL, Journal de Mathematiques, series 3, vol. 8 (1882). PICARD, Annales de l'Ecole Normale, series 2, vol. 10 (1881).

The generator g of S which passes through P contains also the point P_ρ, so that

$$Y = y + l\rho, \tag{69}$$

where l is an arbitrary constant, will represent an arbitrary point Q of g. We shall set up a system of differential equations for Y of form (1), compute its semi-covariants and set up the expression Π for the pinch-point of the Cayley cubic which osculates S at Q, making use of the expression (68).

We begin by deducing a system of differential equations for ρ. We have

$$\begin{aligned} &\rho = y_v, \qquad \rho_u = y_{uv} = \sigma, \qquad \rho_v = y_{vv} = -gy, \\ &\rho_{uu} = y_{uuv} = (2bg - f_v)y - (f + 2b_v)\rho, \\ &\rho_{vv} = y_{vvv} = -g_v y - g\rho, \qquad \rho_{uv} = y_{uvv} = -gz, \end{aligned} \tag{70}$$

whence the system of differential equations for ρ;

$$\rho_{uu} + \frac{2bg - f_v}{g}\rho_v + (f + 2b_v)\rho = 0, \qquad \rho_{vv} - \frac{g_v}{g}\rho_v + g\rho = 0. \tag{71}$$

We have to consider this system together with system (1) for y.

From (69) we find

$$\begin{aligned} &Y_u = z + l\sigma, \qquad Y_v = \rho - lgy, \\ &Y_{uu} = [-f + l(2bg - f_v)]y - [2b + l(f + 2b_v)]\rho, \\ &Y_{vv} = -(g + lg_v)y - lg\rho, \end{aligned} \tag{72}$$

whence

$$\begin{aligned} &Y_{uu} + \frac{l^2(2bg - f_v) + 2b + 2lb_v}{1 + l^2 g}Y_v + \frac{f + lf_v + l^2 g(f + 2b_v)}{1 + l^2 g}Y = 0, \\ &Y_{vv} - \frac{l^2 g_v}{1 + l^2 g}Y_v + \frac{g + l^2 g^2 + lg_v}{1 + l^2 g}Y = 0. \end{aligned} \tag{73}$$

This system is not yet quite of the desired form, in so far as the coefficient of Y_v in the second equation is not equal to zero. In order to transform it into another system which shall have that form, we put

$$Y = \lambda\eta, \qquad \lambda = \sqrt{1 + l^2 g}, \tag{74}$$

which gives

$$\eta_{uu} + 2B\eta_v + F\eta = 0, \qquad \eta_{vv} + G\eta = 0, \tag{75}$$

where

$$B = b + \frac{lb_v - \frac{1}{2}l^2 f_v}{1 + l^2 g},$$

$$F = f + \frac{lf_v + 2l^2gb_v + l^2bg_v}{1 + l^2g} + \frac{l^3g_v(b_v - \frac{1}{2}lf_v)}{(1 + l^2g)^2}, \tag{76}$$

$$G = g + \frac{lg_v + \frac{1}{2}l^2g_{vv}}{1 + l^2g} - \frac{l^4g_v^2}{4(1 + l^2g)^2}.$$

The fundamental semi-covariants of (75) will be

$$\mathfrak{y} = \frac{1}{\lambda}(y + l\rho), \qquad Z = \mathfrak{y}_u = \frac{1}{\lambda}(z + l\sigma),$$

$$P = \mathfrak{y}_v = \frac{1}{\lambda}\left[-\left(lg + \frac{\frac{1}{2}l^2g_v}{1 + l^2g}\right)y + \left(1 - \frac{\frac{1}{2}l^3g_v}{1 + l^2g}\right)\rho\right], \tag{77}$$

$$\Sigma = \mathfrak{y}_{uv} = \frac{1}{\lambda}\left[-\left(lg + \frac{\frac{1}{2}l^2g_v}{1 + l^2g}\right)z + \left(1 - \frac{\frac{1}{2}l^3g_v}{1 + l^2g}\right)\sigma\right].$$

The expression for the pinch-point of the Cayley cubic scroll which osculates S at the point P_y of the generator g is, according to (68),

$$\Pi = -B_uB_v\mathfrak{y} - 4BB_vZ + 2BB_uP + 8B^2\Sigma.$$

The substitution of the above expressions gives

$$\lambda\Pi = x_1y + x_2z + x_3\rho + x_4\sigma, \tag{78}$$

where

$$x_1 = -B_u(B_v + 2BC), \qquad x_2 = -4B(B_v + 2BC),$$
$$x_3 = B_u(-lB_v + 2BD), \qquad x_4 = 4B(-lB_v + 2BD), \tag{79}$$

if we write for abbreviation

$$C = lg + \frac{\frac{1}{2}l^2g_v}{1 + l^2g}, \qquad D = 1 - \frac{\frac{1}{2}l^3g_v}{1 + l^2g}, \tag{80}$$

$$Cl + D = 1 + l^2g.$$

Making use of the integrability conditions, we find

$$B_v + 2BC = b_v + (b_{vv} + 2bg)l,$$
$$-lB_v + 2BD = 2b + b_vl, \tag{81}$$

so that *we obtain*

$$x_1 = -[b_u + b_{uv}l + \tfrac{1}{2}(b_{uvv} + 2gb_u)l^2][b_v + (b_{vv} + 2bg)l],$$
$$x_2 = -[4b + 4b_vl + 2(b_{vv} + 2bg)l^2][b_v + (b_{vv} + 2bg)l], \tag{82}$$
$$x_3 = +[b_u + b_{uv}l + \tfrac{1}{2}(b_{uvv} + 2gb_u)l^2][2b + b_vl],$$
$$x_4 = +[4b + 4b_vl + 2(b_{vv} + 2bg)l^2][2b + b_vl],$$

as the parametric equations of the locus of the pinch-points of the ∞^1 Cayley cubic scrolls which osculate the surface S at the ∞^1 points of one of its generators. We see that the locus is a twisted cubic situated (of course) upon the osculating hyperboloid.

The locus of the nodal lines of the ∞^1 Cayley cubic scrolls must be the osculating hyperboloid H, i. e., its second set of generators. For we have seen that the locus of the nodal lines of the ∞^1 cubic scrolls, which have fourth order contact with S at a fixed point P, is the second set of generators on H. But the osculating Cayley cubic on P is one of these cubic scrolls of fourth order contact, so that its nodal line is one of these generators of the second set on H. As P moves along g, this generator moves on H.

The cubic (82) intersects the generator g of S, whose equations are $x_2 = x_4 = 0$, in two points

$$y + l_k\rho \qquad (k=1, 2),$$

where l_1 and l_2 are the roots of the quadratic

$$2b + 2b_v l + (b_{vv} + 2bg)l^2 = 0,$$

so that

$$l_1 + l_2 = -\frac{2b_v}{b_{vv} + 2bg}, \qquad l_1 l_2 = \frac{2b}{b_{vv} + 2bg}.$$

Consequently

$$(83) \qquad (y + l_1\rho)(y + l_2\rho) = y^2 - \frac{2b_v}{b_{vv} + 2bg} y\rho + \frac{2b}{b_{vv} + 2bg}\rho^2.$$

But this quadratic is precisely the covariant which determines the flecnodes of g. In fact the flecnodes are given by the factors of

$$u'_{12}\rho^2 - u'_{21}y^2 + (u'_{11} - u'_{22})y\rho,$$

where *

$$(84) \qquad \begin{aligned} u'_{11} &= -4f, \qquad u'_{12} = -8b, \\ u'_{21} &= -4f_v + 8bg = 4b_{vv} + 8bg, \\ u'_{22} &= -4f - 8b_v. \end{aligned}$$

It also becomes apparent now why the canonical development breaks down if the point P of the surface S is a flecnode. For, in that case, the pinch-point of the osculating Cayley cubic scroll coincides with P itself, so that the canonical tetrahedron degenerates.

Let us multiply out the terms in (82). If the determinant Δ of the coefficients of l^0, l^1, l^2, l^3 in (82) is different from zero, the twisted cubic will be non-degenerate. We proceed to evaluate that determinant. We have

*Second Memoir, eq. (15).

$$\tfrac{1}{4}\Delta =$$

$$\begin{vmatrix} b_u b_v, & b_u(b_{vv}+2bg)+b_{uv}b_v, & b_{uv}(b_{vv}+2bg)+\tfrac{1}{2}b_v(b_{uvv}+2gb_u), & \tfrac{1}{2}(b_{uvv}+2gb_u)(b_{vv}+2bg) \\ 2bb_v, & 2b(b_{vv}+2bg)+2b_v^2 & 3b_v(b_{vv}+2bg) & (b_{vv}+2bg)^2 \\ 2bb_u & b_u b_v+2bb_{uv} & b_{uv}b_v+b(b_{uvv}+2gb_u) & \tfrac{1}{2}b_v(b_{uvv}+2gb_u) \\ 4b^2 & 6bb_v & 2b_v^2+2b(b_{vv}+2bg) & b_v(b_{vv}+2bg) \end{vmatrix}.$$

Multiply the elements of the second and fourth rows by $-b_u/2b$ and add to those of the first and third respectively; in the resulting determinant multiply the elements of the third and fourth rows by $-b_v/2b$ and add them to the elements of the first and second rows respectively. Making use of the equations which define C' and θ' we find

$$\frac{-\Delta}{4b^2} = \begin{vmatrix} 0 & -\frac{1}{2^{10}}\frac{C'\theta'}{b} & -\frac{1}{2^8}\frac{\theta'}{b}\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right) \\ -\frac{1}{2^6}\theta' & -\frac{1}{2^6}\frac{b_v}{b}\theta' & -\frac{1}{2^7}(b_{vv}+2bg)\frac{\theta'}{b} \\ \frac{1}{2^2}bC' & \frac{1}{2^3}C'b_v+bb_{uvv}-b_u b_{vv} & \frac{1}{2}b_v\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right) \end{vmatrix},$$

whence

$$\Delta = -\frac{\theta'^2}{2^{10}}\left[b^2\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right)^2+\frac{1}{2^5}C'^2 b(b_{vv}+2bg) - \frac{1}{2^2}bb_v C'\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right)\right]. \tag{85}$$

But we have

$$\theta' = 2^6(b_v^2-2bb_{vv}-4b^2 g),$$

$$\theta'_u = 2^6(2b_v b_{uv}-2b_u b_{vv}-2bb_{uvv}-8bb_u g),$$

whence

$$b\theta'_u-2b_u\theta' = 2^7\left[-b^2\left(b_{uvv}-\frac{b_u b_{vv}}{b}\right)+\frac{1}{2^3}bb_v C'\right],$$

so that

$$\Delta = \frac{\theta'^2\mathfrak{B}}{2^{20}b^2}, \tag{86}$$

where

$$\mathfrak{B} = -2^6[(b\theta'_u-2b_u\theta')^2-4b^2C'^2\theta'] \tag{87}$$

is the invariant of the linear complex which osculates the ruled surface S along the generator g.* The cubic curve (82) can therefore be degenerate, only if either the two flecnodes of g coincide, or if the osculating linear complex is special.

*Second Memoir, eq. (47).

Of the problems to be considered in this paragraph there still remains the third one; what is the developable generated by the singular tangent plane of the osculating Cayley cubic scroll as P moves along g? But the answer to this question is very simple. Since the singular tangent plane is tangent to the osculating hyperboloid at the pinch-point of the osculating Cayley cubic scroll, its coördinates (u_1, u_2, u_3, u_4) will be proportional to $(x_4, -x_3, -x_2, x_1)$ where the quantities x_k are defined by equations (82). This developable is therefore circumscribed about the osculating hyperboloid, and its edge of regression is again a space cubic. We may recapitulate the main results of this paragraph in the following theorem.

At every point P of a generator g of a ruled surface S, whose osculating hyperboloid H does not hyperosculate it, there exists a unique osculating Cayley cubic scroll. As P moves along the generator g, the locus of the nodal line of the osculating Cayley cubic scroll consists of those generators of H which intersect g. The locus of the pinch-points is a twisted cubic curve C_1 on H, which passes through the two flecnodes of g and which does not degenerate, except when either the two flecnodes of g coincide, or when the linear complex which osculates S along g becomes special. The locus of the singular tangent planes of the osculating Cayley cubic scrolls is a developable which is circumscribed about the osculating hyperboloid H along the cubic curve C_1, and whose cuspidal edge is another twisted cubic curve C_2 which does not degenerate unless C_1 does also.

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INVARIANTS OF THE FUNCTION $F(x, y, x', y')$ IN THE CALCULUS OF VARIATIONS*

BY

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Introduction.

In KNESER's *Lehrbuch der Variationsrechnung*, § 16, and in BOLZA's *Lectures on the Calculus of Variations*, § 35, the transformation of the definite integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt$$

by a "point transformation,"

$$x = X(u, v), \qquad y = Y(u, v),$$

is incidentally considered, and the invariance of the expressions

$$F_1(x, y, x', y') = \frac{F_{x'x'}}{y'^2} = \frac{F_{x'y'}}{-x'y'} = \frac{F_{y'y'}}{x'^2},$$

$$T(x, y, x', y', x'', y'') = F_{xy'} - F_{yx'} + F_1(x'y'' - x''y'),$$

$$E(x, y, x', y', \bar{x}, \bar{y}) = F(x, y, \bar{x}, \bar{y}) - [\bar{x}F_{x'}(x, y, x', y') + \bar{y}F_{y'}(x, y, x', y')],$$

is proved. The object of the present paper is to study this transformation more in detail, and to derive further invariants which are of importance for the Calculus of Variations.†

In Chapter I the general definitions concerning invariants of the function $F(x, y, x', y')$ with respect to point transformations are given, and a process (which does not essentially differ from the "δ-process" of the Calculus of Variations) is developed, which transforms an absolute invariant again into an absolute invariant. This permits us to derive from known invariants new invariants.

In Chapter II, after a short discussion of the invariants arising out of the first variation, a new absolute invariant

$$K(x, y, x', y', x'', y'', x''', y''')$$

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† See also a paper by G. LANDSBERG, *Krümmungstheorie und Variationsrechnung*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 16, 1907.

connected with the second variation is obtained by means of the process described in Chapter I. When the curve for which the invariant K is computed is in particular an extremal, the invariant K takes the simple form

$$\bar{K} = \frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1}.$$

In Chapter III a transformation of the parameter t of the curve is combined with the point transformation, and by a proper modification of the invariant K a function

$$K_0(x, y, x', y', x'', y'', x''', y''')$$

is obtained which remains invariant not only under every point transformation but at the same time under every parameter transformation. For the case of an extremal the expression K_0 reduces to

$$\bar{K}_0 = \frac{1}{F^2}\left[\bar{K} - \frac{1}{2}\left(\frac{F'''}{F} - \frac{3}{2}\frac{F'^2}{F^2}\right)\right].$$

This leads to an invariantive normal form of the second variation, viz.

$$\delta^2 I = \frac{\epsilon^2}{2}\int_{\alpha_0}^{\alpha_1}\left[\left(\frac{dV}{d\alpha}\right)^2 - \bar{K}_0 V^2\right]d\alpha.$$

In Chapter IV these results are applied to the case of the geodesics, and it is shown that the invariant $\bar{K}_0$ is in this case identical with the Gaussian curvature K; while the general invariant K_0 is expressible in terms of $1/\rho$, the Gaussian curvature of the surface, and of $1/\rho_g$, the geodesic curvature of the curve for which the value of K_0 is taken, by means of the formula

$$K_0 = \frac{1}{\rho} - \frac{1}{2\rho_g^2}.$$

Finally, the result concerning the second variation is applied to JACOBI's theorem on the conjugate points of geodesics on surfaces of negative curvature.

CHAPTER I.* *General theory of the invariants of the calculus of variations for the simplest type of problems.*

§ 1. *Extended point transformations.*

We consider a point transformation connecting two systems of rectangular coördinates,

(1) $$x = X(u, v), \qquad y = Y(u, v).$$

* This chapter is based upon a series of lectures given by Professor BOLZA in his Seminar in the Calculus of Variations during the Spring Quarter, 1905, at the University of Chicago.

For simplicity we suppose that the functions $X(u, v)$, $Y(u, v)$ are analytic functions of u, v, regular in the domain under consideration. Further we assume that the Jacobian

$$D(u, v) \equiv \frac{\partial(X, Y)}{\partial(u, v)}$$

is different from zero in this domain, thus insuring a one-to-one correspondence between every sufficiently small part of this domain and its image in the xy-plane.* The inverse of the transformation (1) we denote by

$$u = U(x, y), \qquad v = V(x, y). \tag{1a}$$

We consider now a curve in the uv-plane

$$L: \qquad u = \phi(t), \qquad v = \psi(t),$$

which we suppose to be regular † in a certain interval (t_0, t_1). Let its image in the xy-plane be

$$L': \quad \begin{cases} x = X[\phi(t), \psi(t)] = \Phi(t), \\ y = Y[\phi(t), \psi(t)] = \Psi(t). \end{cases} \tag{2}$$

Differentiating (2) with respect to t, and denoting derivatives with respect to t by accents, we obtain

$$x' = X_u u' + X_v v', \qquad y' = Y_u u' + Y_v v', \tag{3}$$

$$\begin{aligned} x'' &= X_{uu} u'^2 + 2X_{uv} u'v' + X_{vv} v'^2 + X_u u'' + X_v v'', \\ y'' &= Y_{uu} u'^2 + 2Y_{uv} u'v' + Y_{vv} v'^2 + Y_u u'' + Y_v v'', \end{aligned} \tag{4}$$

and so on.

The totality of all point transformations (1) form a group, T, which in LIE's terminology is an infinite, ‡ continuous group. § If we combine (1) and (3) and consider u', v', x', y', not as derivatives of certain functions with respect to t, but as new variables, the combination (1) and (3) represents an "extended point-transformation" ‖ between the variables, u, v, u', v', on the one hand, and the variables x, y, x', y', on the other hand. It is easily proved that the totality of transformations (1), (3) forms a group, which we denote by T'. Similarly

* OSGOOD, *Lehrbuch der Functionentheorie*, vol. 1, p. 56.

† I. e., $\phi(t)$, $\psi(t)$ are analytic functions of t, regular in (t_0, t_1), and $\phi'(t)$, $\psi'(t)$ do not vanish simultaneously in (t_0, t_1). Compare KNESER, *Lehrbuch der Variations-Rechnung*, p. 3.

‡ "Infinite" inasmuch as the transformation (1) contains the arbitrary functions $X(u, v)$, $Y(u, v)$.

§ Compare LIE, *Die Grundlagen für die Theorie der unendlichen continuirlichen Gruppen*, Leipziger Berichte, 1891, p. 316, and LIE-SCHEFFERS, *Continuirliche Gruppen*, p. 764.

‖ Compare LIE-SCHEFFERS, *Geometrie der Berührungs-Transformationen*, p. 12. It must, however, be remarked that LIE assumes the curve in the non-parametric form $v = f(u)$, and accordingly obtains (3) in non-homogeneous form.

by adjoining further the equations (4) to T', we would obtain a "twice extended point transformation," and a corresponding group T'', and so on.

§ 2. *Definition of invariants of the function* $F(x, y, x', y')$.

We now apply the group T' to a function $F(x, y, x', y')$. We suppose that F is an analytic function of its four arguments, regular in the vicinity of every point

$$x = a, \qquad y = b, \qquad x' = a', \qquad y' = b',$$

for which (a, b) lies in a certain region R of the xy-plane, while at the same time $(a', b') \neq (0, 0)$. We suppose further that F satisfies the homogeneity condition of the calculus of variations,* namely,

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y')$$

for every positive κ. Substituting for x, y, x', y' their values from (1), (3) in our function $F(x, y, x', y')$, we obtain

$$F(x, y, x', y') = F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v'),$$

or

$$F(x, y, x', y') = G(u, v, u', v'), \tag{5}$$

where G is defined by

$$G(u, v, u', v') = F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v').$$

From this and the homogeneity condition, it follows at once that

$$G(u, v, \kappa u', \kappa v') = \kappa G(u, v, u', v')$$

for every positive κ.

From (5) we can compute the partial derivatives of G, for instance,

$$G_{u'} = F_{x'} X_u + F_{y'} Y_u, \qquad G_{v'} = F_{x'} X_v + F_{y'} Y_v, \tag{6}$$

whence

$$F_{x'} = \frac{1}{D}(Y_v G_{u'} - Y_u G_{v'}), \qquad F_{y'} = \frac{1}{D}(-X_v G_{u'} + X_u G_{v'}). \tag{7}$$

Following the general method outlined by LIE in his paper *Ueber Differential-Invarianten*,† we now adjoin equations (7) to (1) and (3), considering $F_{x'}$, $F_{y'}$, $G_{u'}$, $G_{v'}$ as new variables. We obtain then a transformation between the variables $x, y, x', y', F_{x'}, F_{y'}$, on the one hand, and $u, v, u', v', G_{u'}, G_{v'}$, on the other. The totality of these transformations corresponding to the totality of point transformations (1) form a group T_1'.

* Compare KNESER, *Lehrbuch der Variationsrechnung*, § 3; BOLZA, *Lectures on the Calculus of Variations*, § 24 *b*.

† Mathematische Annalen, vol. 24 (1884), p. 537 ff., and in particular p. 569.

We might also extend T_1' by adjoining the solutions F_x, F_y of the equations

$$G_u = F_x X_u + F_y Y_u + F_{x'}(X_{uu}u' + X_{uv}v') + F_{y'}(Y_{uu}u' + Y_{uv}v'),$$

$$G_v = F_x X_v + F_y Y_v + F_{x'}(X_{uv}u' + X_{vv}v') + F_{y'}(Y_{uv}u' + Y_{vv}v'),$$

and still further by adjoining equations for the second partial derivatives of F and G, and so on. In this manner we should finally get an infinite continuous group $T_m^{(\mu)}$, whose transformations connect x, y, x', y', $\cdots$, $x^{(\mu)}$, $y^{(\mu)}$, F_x, F_y, $F_{x'}$, $F_{y'}$, $\cdots$, $F_{y'^{(m)}}$ with u, v, u', v', $\cdots$, $u^{(\mu)}$, $v^{(\mu)}$, G_u, G_v, $G_{u'}$, $G_{v'}$, $\cdots$, $G_{v'^{(m)}}$ where m is the order of the highest partial derivative of F and G.

We next define *Invariants* under these infinite continuous groups as follows. Let $I_F(x, y, x', y', x'', y'', \cdots, x^{(\mu)}, y^{(\mu)})$ be a function of the arguments indicated, and of F and its partial derivatives up to those of the m-th order. Further let $I_G(u, v, u', v', u'', v'', \cdots)$ be the *same* function of $u, v, u', v', u'', v'', \cdots, u^{(\mu)}, v^{(\mu)}$, and of G and its partial derivatives up to those of the m-th order. Then if $I_G(u, v, u', v', \cdots) = I_F(x, y, x', y', \cdots)$ for every transformation of the group $T_m^{(\mu)}$, we say I_F is an *absolute Invariant* * under $T_m^{(\mu)}$. We call I_F an *Invariant of Index* ρ if $I_G(u, v, u', v', \cdots) = D^\rho I_F(x, y, x', y', \cdots)$, where

$$D = \frac{\partial(X, Y)}{\partial(u, v)}.$$

Furthermore call μ, which is the highest order of the derivatives with respect to t occurring in I_F, *the order of the invariant*, and the highest order, m, of the partial derivatives of F, which occurs in I_F, *the class of the invariant*. As examples, we mention

$$G_1(u, v, u', v') = D^2 F_1(x, y, x', y') \dagger \tag{8}$$

of class 2, order 1, index 2, and

$$T_G(u, v, u', v', u'', v'') = D T_F(x, y, x', y', x'', y'') \tag{9}$$

of class 2, order 2, index 1, where

$$F_1 = \frac{F_{x'x'}}{y'^2} = -\frac{F_{x'y'}}{x'y'} = \frac{F_{y'y'}}{x'^2}, \qquad G_1 = \frac{G_{u'u'}}{v'^2} = -\frac{G_{u'v'}}{u'v'} = \frac{G_{v'v'}}{u'^2},$$

$$T_F = F_{xy'} - F_{yx'} + F_1(x'y'' - x''y'), \qquad T_G = G_{uv'} - G_{vu'} + G_1(u'v'' - u''v'),$$

thus showing the invariance of the so-called LEGENDRE and EULER conditions of the calculus of variations under extended point transformations.

* "Differential Invariant" in LIE's terminology.

† BOLZA, loc. cit., p. 183.

§ 3. *Invariants with several sets of cogredient variables.*

We now introduce a second series of variables

$$x`, y`, x``, y``, \cdots, \qquad u`, v`, u``, v``, \cdots$$

cogredient with

$$x', y', x'', y'', \cdots, \qquad u', v', u'', v'', \cdots,$$

that is, connected by the same transformations,

$$x` = X_u u` + X_v v`, \qquad y` = Y_u u` + Y_v v`, \tag{10}$$

and so on. We thus obtain a further extension of our group and corresponding invariants of F, with two sets of cogredient variables, satisfying for all transformations of the group the equation,

$$I_G(u, v, u', v', \cdots, u`, v`, \cdots) = D^p I_F(x, y, x', y', \cdots, x`, y`, \cdots).$$

An example * of such an invariant is

$$x`F_{x'}(x, y, x', y') + y`F_{y'}(x, y, x', y'),$$

for which

$$u`G_{u'}(u, v, u', v') + v`G_{v'}(u, v, u', v') = x`F_{x'}(x, y, x', y') + y`F_{y'}(x, y, x', y'), \tag{11}$$

as may be verified at once by means of (6) and (10).

From (11) it follows at once that the WEIERSTRASS E-function

$$E_F(x, y, x', y', x`, y`) = F(x, y, x`, y`) - [x`F_{x'}(x, y, x', y') + y`F_{y'}(x, y, x', y')]$$

is an absolute invariant, i. e.,

$$E_G(u, v, u', v', u`, v`) = E_F(x, y, x', y', x`, y`).$$

Invariants of this kind also occur when we consider two curves,

$$L: \qquad x = x(t), \qquad y = y(t),$$

and

$$\tilde{L}: \qquad x = \tilde{x}(\tau), \qquad y = \tilde{y}(\tau).$$

At the point of intersection P we have

$$x = \tilde{x}, \qquad y = \tilde{y},$$

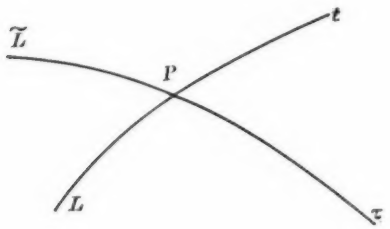
and if we put at P

$$x' = \frac{dx}{dt}, \qquad y' = \frac{dy}{dt}, \qquad x` = \frac{d\tilde{x}}{d\tau}, \qquad y` = \frac{d\tilde{y}}{d\tau},$$

equation (11) shows that the condition of transversality of L and $\tilde{L}$ is preserved in passing from the xy-plane to the uv-plane.

Another invariant of this kind is the expression $x'y` - x`y'$, for which

* BOLZA, loc. cit., p. 183, eq. 28.



(12) $$(u'v` - u`v') = D^{-1}(x'y` - x`y').$$

This last example has an important application in the calculus of variations. When we consider a set of curves in the uv-plane

$$u = u(t, a), \qquad v = v(t, a),$$

then the corresponding set in the xy-plane is

(13) $$x = x(t, a), \qquad y = y(t, a),$$

where

$$x(t, a) = X[u(t, a), v(t, a)], \qquad y(t, a) = Y[u(t, a), v(t, a)].$$

Hence,

$$\frac{\partial x}{\partial t} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial t}, \qquad \frac{\partial x}{\partial a} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial a},$$

$$\frac{\partial y}{\partial t} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial t}, \qquad \frac{\partial y}{\partial a} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial a}.$$

The variables

$$x' = \frac{\partial x}{\partial t}, \qquad y' = \frac{\partial y}{\partial t}, \qquad u' = \frac{\partial u}{\partial t}, \qquad v' = \frac{\partial v}{\partial t},$$

are therefore cogredient with

$$x` = \frac{\partial x}{\partial a}, \qquad y` = \frac{\partial y}{\partial a}, \qquad u` = \frac{\partial u}{\partial a}, \qquad v` = \frac{\partial v}{\partial a},$$

and consequently from (12) the *Jacobian*

$$\Delta(t, a) = \frac{\partial(x, y)}{\partial(t, a)}$$

is an *invariant of index* -1; i. e.,

$$\frac{\partial(u, v)}{\partial(t, a)} = D^{-1}\frac{\partial(x, y)}{\partial(t, a)}.$$

If, in particular, the set of curves (13) is the set of extremals through a fixed point P_0, then the equation

$$\Delta(t, a_0) = 0$$

furnishes the conjugate* point P_0' to P_0 on the particular extremal $a = a_0$. But on account of (9) the image of the extremals in the uv-plane is the set of extremals through the image Q_0 of P_0.

It follows, since Δ is invariant, *that the conjugate Q_0' of Q_0 on the extremal $a = a_0$ in the uv-plane, is the image of the point P_0'. In other words, the conjugate of the image is the image of the conjugate.*†

Another result from (12) is as follows: If $I_F(x, y, x', y', \cdots)$ be an invariant of index ρ, then

$$(x'y` - x`y')^\rho I_F(x, y, x', y' \cdots)$$

is an *absolute invariant.* For instance, $F_1(x'y` - x`y')^2$ is an absolute invariant, according to (8).

The following lemma will be of importance:

From every invariant which contains only the variables $x`$, $y`$ of the second set, and which is, moreover, homogeneous in $x`$, $y`$, an invariant with only one set of variables can be derived by replacing $x`$, $y`$ by $F_{y'}$, $-F_{x'}$, respectively.

Proof. From (7) we have

$$D(F_{y'}) = (G_{v'})X_u + (-G_{u'})X_v, \qquad D(-F_{x'}) = (G_{v'})Y_u + (-G_{u'})Y_v.$$

Hence the equations (10) are satisfied by

$$u` = G_{v'}, \qquad v` = -G_{u'}, \qquad x` = DF_{y'}, \qquad y` = -DF_{x'}.$$

If, therefore,

$$I_G(u, v, u', v', \cdots, u`, v`) = D^\rho I_F(x, y, x', y', \cdots, x`, y`),$$

for all transformations of the group under consideration, then we have in particular,

$$I_G(u, v, u', v', \cdots, G_{v'}, -G_{u'}) = D^\rho I_F(x, y, x', y', \cdots, DF_{y'}, -DF_{x'}).$$

If, furthermore, I is homogeneous in $x`$, $y`$ of degree m, then

$$I_G(u, v, u', v', \cdots, G_{v'}, -G_{u'}) = D^{\rho+m} I_F(x, y, x', y', \cdots, F_{y'}, -F_{x'}),$$

and I is an invariant with only one set of variables. The index is increased by the degree m of the homogeneity of I in $x`$, $y`$.

§ 4. *Methods for the construction of invariants.*

For the determination of all invariants of a given group $T^{(\mu)}_{(m)}$ one might use the general method developed by LIE in his paper *Ueber Differential-Invari-*

* BOLZA, loc. cit., p. 63.

† First given by A. L. UNDERHILL in a paper before the Chicago Section, April, 1905.

anten,* which reduces the problem to the solution of a complete system of linear partial differential equations.† The application of this method to the present problem becomes, however, very complicated on account of the great number of adjoined variables. We therefore employ different methods, which lead more directly to those invariants which are of paramount importance in the calculus of variations.

The simplest of these invariants have been used already as examples in the preceding sections. They are the functions

$$F(x, y, x', y'), \quad x'F_{x'}(x, y, x', y') + y'F_{y'}(x, y, x', y'), \quad F_1(x, y, x', y'),$$
$$T(x, y, x', y', x'', y''), \quad E(x, y, x', y', x', y'), \quad \Delta(t, a).$$

In order to obtain further invariants, we now develop methods for the derivation of invariants from already known invariants. We denote by $I_F(x, y, x', y', \cdots)$ an absolute invariant under $T^{(\mu)}_{(m)}$, so that

$$I_G(u, v, u', v', \cdots) = I_F(x, y, x', y', \cdots) \tag{14}$$

for every transformation of $T^{(\mu)}_{(m)}$.

a) The method of differentiation.

Consider now u, v as functions of t, and $u^{(k)}, v^{(k)}$ as their k-th derivatives with respect to t. Then x, y will be functions of t determined by (2), and similarly their derivatives $x', y', x'', y'', \cdots$ by (3), (4) and the successive derived equations. The equation (14) becomes then an identity in t, and it may be differentiated with respect to t. We have

$$\frac{dI_G}{dt} = \frac{dI_F}{dt}, \tag{15}$$

i. e.,

$$\frac{\partial I_G}{\partial u}u' + \frac{\partial I_G}{\partial v}v' + \frac{\partial I_G}{\partial u'}u'' + \frac{\partial I_G}{\partial v'}v'' + \cdots = \frac{\partial I_F}{\partial x}x' + \frac{\partial I_F}{\partial y}y' + \frac{\partial I_F}{\partial x'}x'' + \frac{\partial I_F}{\partial y'}y'' + \cdots, \tag{16}$$

where it will be remembered that I_F is a function of $x, y, x', y', \cdots$ and of F and some of its partial derivatives which are themselves functions of x, y, x, y'. Equation (16) appears at first as an identity in t. But it holds for *any* functions u, v having derivatives of sufficiently high order. Let $u_0, v_0, u_0', v_0', u_0'', v_0'', \cdots$ be an arbitrary system of values of the variables $u, v, u', v', u'', v'', \cdots$, and $x_0, y_0, x_0', y_0', x_0'', y_0'', \cdots$ the corresponding values of $x, y, x', y', x'', y'', \cdots$ ob-

*Mathematische Annalen, vol. 24 (1884), p. 537.

†This method was used by ZORAWSKI for the geodesic problem; Acta Mathematica, vol. 16 (1892), p. 1.

tained by the transformations (1), (3), (4), and so on. Then in order to show that (16) holds for this arbitrarily chosen system of values, we choose for the functions $u(t)$, $v(t)$ which we substitute in (15),

$$u(t) = u_0 + u_0' \frac{t}{1!} + u_0'' \frac{t^2}{2!} + \cdots + u_0^{(\mu)} \frac{t^{(\mu)}}{\mu!},$$

$$v(t) = v_0 + v_0' \frac{t}{1!} + v_0'' \frac{t^2}{2!} + \cdots + v_0^{(\mu)} \frac{t^{(\mu)}}{\mu!},$$

and put after the differentiation $t = 0$. The right side of (16) is therefore a *new absolute invariant*, whose class and order are each one unit higher than that of I_F, and it accordingly corresponds to a further extended group. Thus we obtain from the absolute invariant,

$$I = F = x' F_{x'} + y' F_{y'}^*$$

of order 1 and class 1, the new absolute invariant

$$x' F_x + y' F_y + x'' F_{x'} + y'' F_{y'}$$

which is of order 2 and class 2.

b) The δ-process.

We suppose now that the functions $u(t)$, $v(t)$ considered under (*a*) depend upon a parameter ϵ. Then equation (14) is an identity in t and ϵ, and we may therefore differentiate it with respect to ϵ,

$$\frac{\partial I_G}{\partial \epsilon} = \frac{\partial I_F}{\partial \epsilon},$$

i. e., if we denote differentiation with respect to ϵ by a dot,

$$(17)\quad \frac{\partial I_G}{\partial u}\dot{u} + \frac{\partial I_G}{\partial v}\dot{v} + \frac{\partial I_G}{\partial u'}\dot{u}' + \frac{\partial I_G}{\partial v'}\dot{v}' + \cdots = \frac{\partial I_F}{\partial x}\dot{x} + \frac{\partial I_F}{\partial v}\dot{v} + \frac{\partial I_F}{\partial x'}\dot{x}' + \frac{\partial I_F}{\partial y'}\dot{y}' + \cdots.$$

This equation appears again at first as an identity in t, ϵ. But since $u(t, \epsilon)$, $v(t, \epsilon)$ are arbitrary, (17) remains true for all systems of values of

$$u, v, u', v', \cdots, \qquad \dot{u}, \dot{v}, \dot{u}', \dot{v}', \cdots,$$

on the one hand, and of

$$x, y, x', y', \cdots, \qquad \dot{x}, \dot{y}, \dot{x}', \dot{y}', \cdots,$$

on the other hand, which are connected by (1), (3), (4) and so on, and also by the following relations obtained from (1), (2), (4) by differentiation with respect to ϵ:

* BOLZA, *Lectures*, p. 120, eq. 9.

$$\dot{x} = X_u\dot{u} + X_v\dot{v}, \qquad \dot{y} = Y_u\dot{u} + Y_v\dot{v},$$
$$\dot{x}' = X_{uu}\dot{u}u' + X_{uv}(\dot{u}v' + u'\dot{v}) + X_{vv}\dot{v}v' + X_u\dot{u}' + X_v\dot{v}',$$
$$\dot{y}' = Y_{uu}\dot{u}u' + Y_{uv}(\dot{u}v' + u'\dot{v}) + Y_{vv}\dot{v}v' + Y_u\dot{u}' + Y_v\dot{v}'.$$

The proof of this last statement is entirely analogous to the corresponding proof given under (*a*). The right hand side of (17) is therefore an absolute invariant under a further extended group obtained by means of these last equations by adjoining the new variables $\dot{x}, \dot{y}, \dot{x}', \dot{y}', \cdots$.

Our process transforms therefore an absolute invariant into an absolute invariant, under a further extended group. We shall call this process the "δ-process," because it is essentially identical with the δ-process of the calculus of variations,* and we shall use, in the sequel, the symbol δ to indicate the first order term of an expansion with respect to a parameter ϵ. Thus

$$\delta I_F = \dot{I}_F\epsilon = \frac{\partial I_F}{\partial \epsilon}\epsilon.$$

Similarly

$$\delta x = \dot{x}\epsilon = \frac{\partial x}{\partial \epsilon}\epsilon, \qquad \delta x' = \dot{x}\,\epsilon = \frac{\partial^2 x}{\partial \epsilon\,\partial t}\epsilon,$$

and so on.

If we have found by the "δ-process" an absolute invariant with respect to the group extended by the adjunction of

$$\dot{x}, \dot{y}, \dot{x}', \dot{y}', \cdots,$$

we may apply the process a second time, and thus obtain a new absolute invariant with respect to the group obtained by adjoining the higher derivatives with respect to ϵ,

$$\ddot{x}, \ddot{y}, \ddot{x}', \ddot{y}', \cdots.$$

Numerous examples of this process will occur in the next chapter.

CHAPTER II. *Invariants under extended point transformations.*

§ 5. *The invariants arising from the first variation.*

We apply the general principles of Chapter I to the invariants connected with the first and second variations of

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt.$$

The starting point is the fact that F is an absolute invariant (cf. (5)) for the

* BOLZA, *Lectures*, § 4c. If we put after the differentiation $\epsilon = 0$, and then multiply by ϵ, we obtain exactly the δ-process of the calculus of variations.

group of extended transformations (1), (3); whence it follows by § 4 that δF and $\delta^2 F$ are also absolute invariants under the same group. We shall be able to break up each of these into invariants which are directly connected with the minimizing of I.

The first variation of F may be thrown into the two forms*

$$\delta F = \delta x\left(F_x - \frac{d}{dt}F_{x'}\right) + \delta y\left(F_y - \frac{d}{dt}F_{y'}\right) + \frac{d}{dt}(F_{x'}\delta x + F_{y'}\delta y)$$

$$= Tw + \frac{d}{dt}(F_{x'}\delta x + F_{y'}\delta y),$$

where

$$w = y'\delta x - x'\delta y.$$

Since δx and δy are cogredient with x' and y', it follows from (11) that

$$F_{x'}\delta x + F_{y'}\delta y$$

is an absolute invariant; and also, by § 4a, its derivative with respect to t. Hence it follows, since δF is an absolute invariant, that Tw must also be an absolute invariant under the extended group of transformations (1), (3). We define $\delta_1 F$ and $\delta_2 F$ by the equations

$$\delta_1 F = \delta x\left(F_x - \frac{d}{dt}F_{x'}\right) + \delta y\left(F_y - \frac{d}{dt}F_{y'}\right) = Tw, \tag{18}$$

$$\delta_2 F = \delta x F_{x'} + \delta y F_{y'},$$

$\delta_1 F$ and $\delta_2 F$ each being an absolute invariant.

§ 6. *First form of* $\delta(\delta_1 F)$.

The work will be similar to the WEIERSTRASS transformation of the second variation,† but will be more general inasmuch as we do not assume, as WEIERSTRASS does, that the second variation is computed along an extremal. We define

$$L = F_{xx'} - y'y''F_1, \quad M = \tfrac{1}{2}[F_{xy'} + F_{yx'} + F_1(x'y'' + x''y')], \quad N = F_{yy'} - x'x''F_1,$$

whence it follows that

$$Lx' + My' = F_x - \tfrac{1}{2}Ty', \qquad Mx' + Ny' = F_y + \tfrac{1}{2}Tx'.$$

With WEIERSTRASS we set

$$L_1 = F_{xx} - \frac{dL}{dt} - y''^2 F_1,$$

* BOLZA, loc. cit., § 25a.

† BOLZA, loc. cit., § 27a.

(19) $$M_1 = F_{xy} - \frac{dM}{dt} + x''y''F_1,$$

$$N_1 = F_{yy} - \frac{dN}{dt} + x''^2F_1,$$

whence

(20) $$L_1x' + M_1y' = \tfrac{1}{2}T'y', \qquad M_1x' + N_1y' = -\tfrac{1}{2}T'x'.$$

If the "δ-process" is applied to the first form of $\delta_1 F$, this form becomes, after reduction,

(21) $$\delta(\delta_1 F) = T\delta w - w\frac{d}{dt}(F_1 w') + L_1\delta x^2 + 2M_1\delta x\delta y + N_1\delta y^2.$$

§ 7. *Second form of* $\delta(\delta_1 F)$.

In order to apply more conveniently the subscript notation we will place

$$F_1 = H.$$

From the definition (14a) of H follow the equations:

$$F_{x'x'x} = y'^2H_x, \qquad F_{x'y'x} = -x'y'H_x, \qquad F_{y'y'x} = x'^2H_x,$$
$$F_{x'x'y} = y'^2H_y, \qquad F_{x'y'y} = -x'y'H_y, \qquad F_{y'y'y} = x'^2H_y.$$

Futhermore H is positively homogeneous of degree -3 in x' and y', i. e.,

$$H_{x'}x' + H_{y'}y' = -3H.$$

Making use of these relations and applying the "δ-process" to the second form of $\delta_1 F$ in (18), we obtain

(22) $$\delta(\delta_1 F) = w\left[A\delta x + B\delta y - \frac{d}{dt}(w'H)\right] + T\delta w,$$

where

(23) $$A = T_x + \frac{d}{dt}(Hy''), \qquad B = T_y - \frac{d}{dt}(Hx'').$$

A comparison of (21) and (22) then gives

(24) $$L_1\delta x^2 + 2M_1\delta x\delta y + N_1\delta y^2 = w(A\delta x + B\delta y).$$

§ 8. *The absolute invariant* $\Phi(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y)$.

The absolute invariant $\delta(\delta_1 F)$ contains the variations δx, δy, $\delta x'$, $\delta y'$, $\delta x''$, $\delta y''$. The next step is to split $\delta(\delta_1 F)$ into an aggregate of absolute invariants, one of which contains only the variations δx and δy, and these homogeneously. Then by § 3 we shall derive an invariant with only one set of variables, i. e., not containing any δ's.

For this purpose we first introduce in $\delta(\delta_1 F)$ of § 7 the absolute invariant ω defined by the equation $\omega = wH^{\frac{1}{2}}$, and assume $H \neq 0$ along the curve considered. It follows after reduction that

$$\begin{aligned}\delta(\delta_1 F) = {} & \frac{T}{H^{\frac{1}{2}}}\delta\omega - \frac{T}{2H^{\frac{1}{2}}}\omega\frac{d}{dt}\left(\frac{H_{x'}\delta x + H_{y'}\delta y}{H}\right) - \omega\omega'' \\ & - \frac{T}{2H^{\frac{1}{2}}}\omega\left[\frac{H_x\delta x + H_y\delta y}{H} - \delta x\frac{d}{dt}\left(\frac{H_{x'}}{H}\right) - \delta y\frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right] \\ & + \frac{\omega}{H^{\frac{1}{2}}}(A\delta x + B\delta y) - \omega^2\left(\frac{H'^2}{4H^2} - \frac{H''}{2H}\right).\end{aligned} \tag{25}$$

Since T and H are invariants of index -1 and $+2$ respectively, and since ω, and therefore ω'' and $\delta\omega$ are absolute invariants, it can be shown that each of the first three terms of $\delta(\delta_1 F)$ is itself an absolute invariant.

On account of the fact that $\delta(\delta_1 F)$ is an absolute invariant itself, we obtain the following absolute invariant under the extended group (1), (3), which contains only the variations δx and δy, and these *linearly* and *homogeneously*:

$$\begin{aligned}&\Phi(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y) \\ &= \frac{1}{2}\frac{T}{H^{\frac{1}{2}}}\left[-\frac{H_x\delta x + H_y\delta y}{H} + \delta x\frac{d}{dt}\left(\frac{H_{x'}}{H}\right) + \delta y\frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right] \\ &\quad + \frac{1}{H^{\frac{1}{2}}}(A\delta x + B\delta y) - \omega\left(\frac{H'^2}{4H^2} - \frac{H''}{2H}\right).\end{aligned} \tag{26a}$$

With the help of (24) this may be written

$$\begin{aligned}&\omega\Phi(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y) \\ &= \frac{1}{2}\frac{T\omega}{H^{\frac{1}{2}}}\left[-\frac{H_x\delta x + H_y\delta y}{H} + \delta x\frac{d}{dt}\left(\frac{H_{x'}}{H}\right) + \delta y\frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right] \\ &\quad + L_1\delta x^2 + 2M_1\delta x\delta y + N_1\delta y^2 - \omega^2\left(\frac{H'^2}{4H^2} - \frac{H''}{2H}\right).\end{aligned} \tag{26b}$$

§ 9. *The absolute invariant* $K(x, y, x', y', x'', y'', x''', y''')$.

The absolute invariant Φ is homogeneously linear in δx, δy, these latter variables being cogredient with x', y'. Therefore we obtain by § 3 an invariant of index 1, if in Φ we replace δx and δy by $F_{y'}$ and $-F_{x'}$ respectively. In order to obtain an absolute invariant, we divide by $H^{\frac{1}{2}}$ and denote the final result by $-KF$. We have then the following theorem:

THEOREM. *The function* $F(x, y, x', y')$ *possesses with respect to the point transformation*

$$x = X(u, v), \qquad y = Y(u, v),$$

and its extensions, the following absolute invariant:

$$K(x, y, x', y', x'', y'', x''', y''') = -\frac{1}{FH}(AF_{y'} - BF_{x'}) + \left(\frac{H^2}{4H^2} - \frac{1}{2}\frac{H''}{H}\right) - \frac{1}{2}\frac{T}{FH}\left[-F_{y'}\left\{\frac{H_x}{H} - \frac{d}{dt}\left(\frac{H_{x'}}{H}\right)\right\} + F_{x'}\left\{\frac{H_y}{H} - \frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right\}\right]. \tag{27a}$$

This invariant may also be written in the following form:

$$K(x, y, x', y', x'', y'', x''', y''') = -\frac{1}{F^2H}[L_1F_{y'}^2 - 2M_1F_{x'}F_{y'} + N_1F_{x'}^2] + \left(\frac{H^2}{4H^2} - \frac{1}{2}\frac{H''}{H}\right) - \frac{1}{2}\frac{T}{FH}\left[-F_{y'}\left\{\frac{H_x}{H} - \frac{d}{dt}\left(\frac{H_{x'}}{H}\right)\right\} + F_{x'}\left\{\frac{H_y}{H} - \frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right\}\right]. \tag{27b}$$

Up to this point the curve

$$x = x(t), \qquad y = y(t),$$

for which K is computed and to which the differentiations with respect to t refer, has been entirely arbitrary. K will simplify considerably when this curve is an extremal. In this case $T = 0$, and as a result the right hand members of (20) are zero, and hence we may write with WEIERSTRASS

$$L_1 = y'^2F_2, \qquad M_1 = -x'y'F_2, \qquad N_1 = x'^2F_2.$$

The substitution of these in (27*b*) leads to the following result:

Corollary. For the special case of an extremal the invariant K takes the form

$$\bar{K} = \frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1}, \tag{28a}$$

where F_1, F_2 have the same meaning as in the WEIERSTRASS *theory, and the stroke indicates, here and hereafter, that K refers to an extremal.*

Had we used (27*a*), $\bar{K}$ would appear in the form

$$\bar{K} = \frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{\bar{A}F_{y'} - \bar{B}F_{x'}}{FF_1}. \tag{28b}$$

By comparing these last two equations a new expression for the WEIERSTRASS function F_2 is found, viz.:

$$F_2 = \frac{1}{F}(F_{y'}\bar{A} - F_{x'}\bar{B}), \tag{29}$$

where $\bar{A}$ and $\bar{B}$ are the values of A, B along an extremal.

CHAPTER III. *Invariants under combined parameter and extended point transformations. A normal form of* $\delta^2 I$.

§ 10. *Invariants with respect to parameter transformations.*

The principal object of the present chapter is to derive from the invariant K another invariant K_0, which will remain invariant not only under every point transformation but also under every parameter transformation. Let

$$t = \chi(\tau), \tag{30}$$

be an admissible parameter transformation,* and $\tau = \theta(t)$ the inverse transformation. For brevity set

$$\frac{d\tau}{dt} = \theta'(t) = \lambda,$$

where $\lambda > 0$, and denote by $\bar{\phi}$ the transform of a function $\phi(t)$ by means of (30). We are easily led to the following results:

$$F = \lambda \bar{F}, \qquad F dt = \bar{F} d\tau, \qquad H = \lambda^{-3} \bar{H}, \qquad T = \bar{T}.$$

We define $I(x, y, x', y', x'', y'', \cdots)$ as an absolute invariant with respect to the parameter transformation (30), if it satisfies

$$I(x, y, x', y', \cdots) = I(\bar{x}, \bar{y}, \bar{x}', \bar{y}', \cdots),$$

or $I = \bar{I}$, for all systems of the arguments connected by the relations:

$$\begin{aligned} x &= \bar{x}, & y &= \bar{y}, \\ x' &= \lambda \bar{x}', & y &= \lambda \bar{y}', \\ x'' &= \lambda^2 \bar{x}'' + \lambda' \bar{x}', & y &= \lambda^2 \bar{y}'' + \lambda' \bar{y}', \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

where $\lambda > 0$.

If the functions $x(t), y(t)$ depend upon a parameter ϵ, whereas $\chi(\tau)$ of (30) is independent of ϵ, one finds at once that

$$\delta x = \delta \bar{x}, \qquad \delta y = \delta \bar{y},$$

where the operator δ is again equivalent to $\epsilon \cdot \partial / \partial \epsilon$.

Then more generally, if $I(x, y, x', y', \cdots)$ be an absolute invariant with respect to parameter transformation, the equation $I = \bar{I}$ becomes an identity in ϵ and τ, if we substitute in the left hand side $t = \chi(\tau)$. Hence

$$\delta I = \delta \bar{I};$$

i. e., the "δ-process" transforms every absolute invariant with respect to parameter transformations again into an absolute invariant.

* BOLZA, loc. cit., § 24*a*.

It follows that

$$w = \lambda \bar{w}, \qquad \omega = \lambda^{-\frac{1}{2}} \bar{\omega},$$

where

$$\omega = wH^{\frac{1}{2}}.$$

§ 11. *Combined point and parameter transformation.*

To a curve

$$L: \qquad x = x(t), \qquad y = y(t),$$

we now apply first the point transformation (1a) and then the parameter transformation (30), or vice versa, since they are commutable. As a result of the relations found in § 2 and § 10, we find

$$F(x, y, x', y') = G(u, v, u', v') = \lambda G(\bar{u}, \bar{v}, \bar{u}', \bar{v}'),$$

$$F_1(x, y, x', y') = D^{-2} G_1(u, v, u', v') = D^{-2}\lambda^{-3} G_1(\bar{u}, \bar{v}, \bar{u}', \bar{v}'),$$

$$T_F(x, y, x', y', x'', y'') = D^{-1} T_G(u, v, u', v', u'', v'') = D^{-1} T_G(\bar{u}, \bar{v}, \bar{u}', \bar{v}', \bar{u}'', \bar{v}''),$$

$$\omega = \lambda^{-\frac{1}{2}} \bar{\omega},$$

where

$$\bar{\omega} = (\bar{v}'\,\delta\bar{u} - \bar{u}'\,\delta\bar{v}) \cdot \{H(u, v, \bar{u}', \bar{v}')\}^{\frac{1}{2}}.$$

From these last results it follows that the functions

$$S = \frac{T}{H^{\frac{1}{2}} F^{\frac{1}{2}}}, \tag{31}$$

$$V = \omega F^{\frac{1}{2}}, \tag{32}$$

are absolute invariants under the application of combined point and parameter transformations.

§ 12. *The absolute invariant* $\Phi_0(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y)$.

Our starting point is the fact that wT/F is an absolute invariant under the combined group, and that when the "δ-process" is applied to this expression, we again obtain an absolute invariant which in turn may be broken up into several separate absolute invariants. We may write

$$\delta\left(\frac{wT}{F}\right) = \delta(VS) = S\delta V + V\delta S,$$

and will fix our attention on δS, which is an absolute invariant since S, V, and δV are absolute invariants. This has the form

$$\delta S = \delta\left(\frac{T}{H^{\frac{1}{2}} F^{\frac{1}{2}}}\right) = S\left(\frac{\delta T}{T} - \frac{1}{2}\frac{\delta H}{H} - \frac{3}{2}\frac{\delta F}{F}\right).$$

Since S is an absolute invariant and since λ is independent of ϵ, it follows that $\delta F/F$ is an absolute invariant, and we are led to the following expression as an absolute invariant under the combined group:

$$\frac{\delta T}{T}-\frac{1}{2}\frac{\delta H}{H}=\frac{1}{T}\left[A\delta x+B\delta y-H^{1}\left\{\omega''+\omega\left(\frac{H'^{2}}{4H^{2}}-\frac{1}{2}\frac{H''}{H}\right)\right\}\right]$$
$$-\frac{1}{2}\left[\frac{H_x\delta x+H_y\delta y}{H}+\frac{H_{x'}\delta x'+H_{y'}\delta y'}{H}\right]$$

Next we notice that if I is an absolute invariant under the combined group then so is

$$\frac{1}{F}\frac{dI}{dt}.$$

Setting

$$\frac{1}{F}V'=V_1,\qquad \frac{1}{F}V_1'=V_2, \tag{33}$$

in which V_1 and V_2 are each absolute invariants, we may write

$$\frac{\delta T}{T}-\frac{\delta H}{2H}=\frac{1}{T}(A\delta x+B\delta y)+\frac{V}{2SF^{2}}\left[\frac{F'''}{F}-\frac{3}{2}\frac{F'^{2}}{F^{2}}\right]$$
$$-\frac{1}{F^{2}}\frac{V}{S}\left[\frac{H'^{2}}{4H^{2}}-\frac{1}{2}\frac{H''}{H}\right]+\frac{1}{2}\frac{\delta x}{F}\frac{d}{dt}\left[\frac{FH_{x'}}{H}\right]-\frac{1}{2}\left[\frac{H_x\delta x+H_y\delta y}{H}\right]$$
$$+\frac{1}{2}\frac{\delta y}{F}\frac{d}{dt}\left[\frac{FH_{y'}}{H}\right]-\frac{V_2}{S}-\frac{1}{2F}\frac{d}{dt}\left[\frac{F(H_{x'}\delta x+H_{y'}\delta y)}{H}\right].$$

Since H is homogeneous in x', y' of order 3, it follows that the last term and also the next to the last term are absolute invariants. Hence the remaining terms multiplied by S, constitute an absolute invariant under the combined group. Denoting it by Φ_0 we obtain

$$\Phi_0(x,y,x',y',x'',y'',x''',y''',\delta x,\delta y)=P\delta x+Q\delta y$$
$$+\frac{V}{2F^{2}}\left[\frac{F'''}{F}-\frac{3}{2}\frac{F'^{2}}{F^{2}}\right]-\frac{V}{F^{2}}\left[\frac{H'^{2}}{4H^{2}}-\frac{1}{2}\frac{H''}{H}\right],$$

in which

$$P=\frac{SA}{T}-\frac{S}{2}\frac{H_x}{H}+\frac{S'}{2F}\frac{d}{dt}\left[\frac{FH_{x'}}{H}\right],$$

$$Q=\frac{SB}{T}-\frac{S}{2}\frac{H_y}{H}+\frac{S}{2F}\frac{d}{dt}\left[\frac{FH_{y'}}{H}\right].$$

A and B have the meanings given in (23), and S is defined by (31).

§ 13. *The absolute invariant* $K_0(x, y, x', y', x'', y'', x''', y''')$.

The absolute invariant Φ_0 under the combined group is homogeneous and linear in δx and δy, and these variables are cogredient with x', y'. Since the parameter part of the group introduces only powers of λ, and since in the cogredient variables no such factors are possible, we can apply § 3 and get an invariant of index 1 if in Φ_0 we replace δx and δy by $F_{y'}$ and $-F_{x'}$ respectively. In order to obtain an absolute invariant we multiply by $1/H^{\frac{1}{3}}F^{\frac{1}{3}}$ and denote the result by $-K_0$. Noticing that $V=\omega F^{\frac{1}{3}}$ becomes by this substitution $F^{\frac{1}{3}}H^{\frac{1}{3}}$, we obtain thus the theorem:

THEOREM. *The function* $F(x, y, x', y')$ *possesses with respect to the extended point transformation*

$$x = X(u, v), \qquad y = Y(u, v),$$

and the parameter transformation $t=\chi(\tau)$ *the following absolute invariant:*

$$K_0(x, y, x', y', x'', y'', x''', y''')$$

$$= \frac{1}{F^2}\left[K(x, y, x', y', x'', y'', x''', y''') - \frac{1}{2}\left(\frac{F'''}{F} - \frac{3}{2}\frac{F''^2}{F^2}\right)\right] \tag{34}$$

$$+ \frac{T}{2HF^4}\left[\frac{H_{y'}F_{x'} - H_{x'}F_{y'}}{H}\right]F',$$

where K is given by (27a).

As before in § 9, so here in this chapter we have used as yet a general curve

$$x = x(t), \qquad y = y(t),$$

for which K_0 has been computed, and to which the differentiation with respect to t referred. In case the curve is an extremal along which $T=0$ the formula (34) becomes

$$\overline{K}_0 = \frac{1}{F^2}\left[\frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1} - \frac{1}{2}\frac{F'''}{F} + \frac{3}{4}\frac{F''^2}{F^2}\right]. \tag{35}$$

§ 14. *Computation of* $\overline{K}_0$ *for Kneser's curvilinear coördinates.*

We suppose that the u, v introduced by (1a) instead of x, y are KNESER'S curvilinear coördinates;* in other words, the curves $v=c$ are extremals for the integral

$$I = \int_{t_0}^{t_1} G(u, v, u', v')\,dt,$$

while the curves $u=c$ are the transversals of this set of extremals. The function $G(u, v, u', v')$ has the following characteristic properties:†

$$G(u, v, u', 0) = u', \qquad G_{u'}(u, v, u', 0) = 1, \qquad G_{v'}(u, v, u', 0) = 0.$$

* KNESER, loc. cit., § 16; BOLZA, loc. cit., § 35 *b*, *c*.

† BOLZA, loc. cit., p. 185.

We propose to compute $\overline{K}_0$. The arguments of the functions G, G_1, G_2 are

$$u = u(t), \qquad v = c, \qquad u' = u'(t), \qquad v' = v'(t) = 0,$$

and one can easily compute the values

$$L = 0, \qquad M = 0, \qquad N \doteq -uu''G_1.$$

But since $\overline{K}_0$ is an absolute invariant under the combined group, we may choose the parameter t on the extremal $v = c$ at pleasure. We select $u = \pm t$ according as u increases or decreases with t. Then also

$$N = 0,$$

and from (19)

$$L_1 = M_1 = N_1 = 0,$$

so that

$$G_2 = 0.$$

Further

$$G = u' = \pm 1, \qquad G' = 0, \qquad G'' = 0.$$

Finally if we put

$$G_1(u, v, \pm 1, 0) = h(u, v),$$

then

$$G_1' = \pm h_u, \qquad G_1'' = \pm h_{uu},$$

and substituting these values in (35) we have the following theorem:

THEOREM. *For* KNESER'S *normal form of the integral I the absolute invariant $\overline{K}_0$ takes the following form,*

$$\overline{K}_0 = -\frac{1}{\sqrt{h}} \frac{\partial^2 \sqrt{h}}{\partial u^2}, \tag{36}$$

where $h(u, v) = G_1(u, v, \pm 1, 0)$.

§ 15. *An invariantive normal form of the second variation.*

We restrict now our problem by supposing the end points to be fixed. Then the second variation computed along an extremal of

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

is

$$\overline{\delta^2 I} = \int_{t_0}^{t_1} w \delta T dt.$$

By using the results of §§ 7, 8, 9 this may be transformed into

$$\overline{\delta^2 I} = -\int_{t_0}^{t_1} \omega(\omega'' + \overline{K}\omega)\, dt,$$

and then, by introducing V as defined in (32), into

$$\overline{\delta^2 I} = -\int_{t_0}^{t_1} FV(V_2 + \overline{K}_0 V)dt.$$

In case a change of variable is made by means of

$$\alpha = \int_{t_0}^{t} Fdt,$$

where to t_0 and t_1 correspond α_0 and α_1, we find

$$\overline{\delta^2 I} = -\int_{\alpha_0}^{\alpha_1}\left(\frac{d^2 V}{d\alpha^2} + \overline{K}_0 V\right) Vd\alpha.$$

Integrating the first term by parts, and recalling that V by definition vanishes for α_0 and α_1, we have the result:

When F_1 is positive along the extremal under consideration, the second variation of

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

may be reduced to the invariantive normal form

$$\delta^2 I = \int_{\alpha_0}^{\alpha_1}\left[\left(\frac{dV}{d\alpha}\right)^2 - \overline{K}_0 V^2\right] d\alpha, \tag{37}$$

in which

$$V = \omega F^{\frac{1}{2}}, \qquad \alpha = \int_{t_0}^{t} Fdt,$$

and $\overline{K}_0$ is the invariant defined in § 13.

As a result of this form the following corollary may at once be stated:

Corollary. In case $\overline{K}_0 < 0$ along the extremal we have $\delta^2 I > 0$.

CHAPTER IV. *Application to the geodesic problem.*

§ 16.

The function $F(x, y, x', y')$ for the geodesic problem has the form

$$F(x, y, x', y') = \sqrt{\mathcal{E}x'^2 + 2\mathcal{F}x'y' + \mathcal{G}y'^2},$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the well-known functions of x and y only, used in surface theory. The functions H, F_1 and T have the following values:

$$F_1 = \frac{\mathcal{E}\mathcal{G} - \mathcal{F}^2}{\left(\sqrt{\mathcal{E}x'^2 + 2\mathcal{F}x'y' + \mathcal{G}y'^2}\right)^3}, \qquad T = \frac{\Gamma}{\left(\sqrt{\mathcal{E}x'^2 + 2\mathcal{F}x'y' + \mathcal{G}y'^2}\right)^3}, \tag{38}$$

where

$$\Gamma = (\mathcal{E}\mathcal{G} - \mathcal{F}^2)(x'y'' - x''y')$$
$$+ (\mathcal{E}x' + \mathcal{F}y')\left[(\mathcal{F}_x - \tfrac{1}{2}\mathcal{E}_y)x' + \mathcal{G}_x x'y' + \tfrac{1}{2}\mathcal{G}_y y'^2\right]$$
$$- (\mathcal{F}x' + \mathcal{G}y')\left[\tfrac{1}{2}\mathcal{E}_x x'^2 + \mathcal{E}_y x'y' + (\mathcal{F}_y - \tfrac{1}{2}\mathcal{G}_x)y'^2\right].$$

a) The invariant S.

In the case of a geodesic the absolute invariant

$$S = \frac{T}{H^{\frac{1}{2}}F^{\frac{3}{2}}}$$

is identical with the geodesic curvature.*

b) The invariant $\overline{K}_0$.

Since $\overline{K}_0$ is an absolute invariant for point as well as parameter transformations, we select the Gaussian normal form

$$ds^2 = du^2 + m^2 dv^2,$$

in which u, v are geodesic parallel coördinates, and take for t the arc of the curve on the surface, so that

$$G = \sqrt{u'^2 + m^2 v'^2} = 1.$$

Since this system of coördinates is identical for the geodesic problem with the KNESER coördinates in § 14, we may use the results there found.

From (38) it follows that

$$G_1 = m^2,$$

and by using (36), that

$$\overline{K}_0 = -\frac{1}{m}\frac{\partial^2 m}{\partial u^2}.$$

Accordingly we have the result:

In the case of the geodesic problem the absolute invariant K_0 is identical with the Gaussian curvature.†

Combining this result with (37), we have the JACOBI-BONNET‡ theorem:

On a surface of negative curvature a given point has no conjugate point on the geodesics which pass through it.

* BOLZA, loc. cit., pp. 129, 146.

† SCHEFFERS, *Anwendung der Differential- und Integral-Rechnung auf Geometrie*, vol. 2, p. 503.

‡ Comptes Rendus, vol. 40 (1855), p. 1311; vol. 41 (1855), p. 32. JACOBI, *Gesammelte Werke*, supplementary volume, p. 46.

c) Computation of K_0.

For the computation of K_0 we assume a system of isometric coördinates; i. e.,

$$F = \sqrt{\varepsilon(x'^2 + y'^2)}$$

and substitute this value of F in (34). Since (34) is an invariant under parameter transformation, we may then choose as the parameter t, the arc s of the curve which makes $F = 1$. It will be found that

$$K_0 = -\frac{\varepsilon_{xx} + \varepsilon_{yy}}{2\varepsilon^2} + \frac{\varepsilon_x + \varepsilon_y}{2\varepsilon^3} - \frac{1}{2}\frac{T^2}{\varepsilon^2}. \tag{39}$$

In the case of the geodesics the absolute invariant K_0 has the value

$$K_0 = \frac{1}{\rho} - \frac{1}{2}\left[\frac{1}{\rho_g}\right]^2, \tag{40}$$

where $1/\rho$ is the Gaussian curvature * of the surface at the point (x, y) and $1/\rho_g$ is the geodesic curvature † at the same point.

* SCHEFFERS, loc. cit., vol. 2, p. 493.

† Cf. § 16 *a*), above.

THE INTEGRATION OF A SEQUENCE OF FUNCTIONS AND ITS APPLICATION TO ITERATED INTEGRALS

BY

R. G. D. RICHARDSON*

Introduction.

The fundamental principle involved in this memoir is the integration of a sequence of functions. As this depends essentially upon an inversion in the order of passing to a limit, applications are readily made to several important topics. In particular, the equality of a multiple integral and the corresponding iterated integrals is here treated, also the integration of a series term by term, and the differentiation of both series and definite integrals.

Hitherto in the integration of sequences the condition of uniform convergence in some form has played an important rôle. In this paper the methods are entirely independent of any such condition of convergence. The problem is then, in this respect, brought into more intimate relation with that of the inversion of the order of summation in an absolutely convergent multiple series. The field of integration is taken in § 3 to be a limited m-dimensional region A which either possesses content or is closed. Throughout A there is defined a sequence of functions $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m), \cdots$ converging to the function $f(x_1, \cdots, x_m)$. If $f_1, f_2, \cdots$ form a monotone increasing sequence of integrable positive functions, it is shown that

$$\lim_{n=\infty} \int_A f_n = \underline{\int}_A f.$$

This is included in a more comprehensive result stated in theorem 13.

If for some points of A the functions $f_1, f_2, \cdots$ form a monotone increasing sequence and for the remaining points they form a monotone decreasing sequence, then provided $f, f_1, f_2, \cdots$ are integrable,

$$\lim_{n=\infty} \int_A f_n = \int_A f. \tag{1}$$

* Presented to the Society April 28 and September 3, 1906, and April 27, 1907. Received for publication January 24, 1908.

The equality (1) holds also for a sequence of functions, the totality of which is limited ($|f_i| \leqq G$, $i = 1, 2, \cdots$). Corresponding theorems for differentiation are developed.

The relations

$$\begin{aligned}
&\bar{\int}_T f(x,y)\,dT \geqq \bar{\int}_X dx \bar{\int}_Y f(x,y)\,dy \geqq \bar{\int}_X dx \underline{\int}_Y f(x,y)\,dy \geqq \underline{\int}_T f(x,y)\,dT \\
&\bar{\int}_T f(x,y)\,dT \geqq \underline{\int}_X dx \bar{\int}_Y f(x,y)\,dy \geqq \underline{\int}_X dx \underline{\int}_Y f(x,y)\,dy \geqq \underline{\int}_T f(x,y)\,dT
\end{aligned} \tag{2}$$

are proved in the standard texts for the case that f is a limited function and T a portion of the plane bounded by a simple curve. The most general treatment is that by Professor PIERPONT.* Various simple examples may be given where the existence of one or more of the integrals

$$\int_T f(x,y)\,dT, \qquad \int_X dx \int_Y f(x,y)\,dy, \qquad \int_Y dy \int_X f(x,y)\,dx$$

is not a sufficient condition for the existence of the others.† When the function f has infinite discontinuities, the problem is much more complex. For the case where the discontinuities are arranged on a finite number of regular curves, relations similar to (2) have been derived. Under this restriction, if the double integral and the iterated integral exist, they are equal. The first *general* treatment of the subject is that of DE LA VALLÉE-POUSSIN. The problem was later considered by SCHÖNFLIES, PIERPONT and HOBSON, the most general treatment from some standpoints being that of HOBSON. For a field comprising *all* the points of a *rectangle*, he has shown that if the double integral and the iterated integral of a *positive* function exist they are equal. In this article it will be proved (§ 4) that for *any* function f the existence (finite or infinite) of the integrals

$$\int_T f(x,y)\,dT, \qquad \int_X dx \int_Y f(x,y)\,dy$$

is a sufficient condition for their equality, the only conditions on the field being that T possess content and that X possess content or is closed. In the discussions of this memoir we use the broad definition of integral which allows the existence of non-absolutely convergent single integrals, while Hobson's work is limited to the narrower absolutely convergent integrals. Corresponding to (2) very general relations for an unlimited function f are developed. For example, if $f \geqq -G$, it is shown under the same limitations for T and X that

* *Lectures on the Theory of Functions of Real Variables*, vol. 1, Ginn & Co., Boston (1905). This will be referred to as *Lectures*.

† See p. 361 of this article.

$$\int_T \bar{f}(x, y)\, dT \geqq \underline{\int}_X dx \int_Y \bar{f}(x, y)\, dy \geqq \underline{\int}_X dx \underline{\int}_Y f(x, y)\, dy \geqq \underline{\int}_T f(x, y)\, dT,$$

where one or more of the integrals may be unlimited. An attempt is made in § 4 to discuss the results of DE LA VALLÉE-POUSSIN, SCHÖNFLIES and HOBSON.

In § 5 the results of § 3 are applied to the integration of a convergent series

$$U(x_1, \cdots, x_m) = u_1(x_1, \cdots, x_m) + u_2(x_1, \cdots, x_m) + \cdots.$$

If $u_1, u_2, \cdots$ are positive integrable functions, then

$$\underline{\int}_A U = \int_A u_1 + \int_A u_2 + \cdots.$$

If for some points of A, the functions $u_1, u_2, \cdots$ are all positive or zero, while for the remaining points $u_1, u_2, \cdots$ are all negative or zero, then provided U, $u_1, u_2, \cdots$ are integrable

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots. \tag{3}$$

Relation (3) is true also if for all values of n we have $|\sum_{i=1}^{n} u_i| \leqq G$. These relations are generalized and the corresponding theorems for differentiation developed.

In a later article these topics will be discussed for an infinite field, the relation to the integrals of LEBESGUE will be shown, and the conditions for inversion of order in an iterated integral will be treated.

§ 1. *Preliminary notions and definitions.*

Let $\mathfrak{A}$ denote a limited aggregate in m-dimensional space. We may effect a division of space into cells each of which has its greatest dimension equal to or less than δ. This is indicated by saying that the division is of *norm* δ. When δ approaches zero, the sum of those cells containing at least one point of $\mathfrak{A}$ approaches a limit which is called the *upper content* of $\mathfrak{A}$ and is written $\bar{\mathfrak{A}}$. In a similar manner we designate by $\underline{\mathfrak{A}}$ the *lower content* of $\mathfrak{A}$, which is the limit of the sum of those cells all of whose points belong to $\mathfrak{A}$. If $\bar{\mathfrak{A}} = \underline{\mathfrak{A}}$, the aggregate is said to possess *content* $\mathfrak{A}$; if $\bar{\mathfrak{A}} = 0$, it is called *discrete*. The *frontier* of $\mathfrak{A}$ consists of an aggregate F such that in the infinitesimal vicinity of each point of F there is at least one point of $\mathfrak{A}$ and one not of $\mathfrak{A}$. If $\mathfrak{A}$ possesses content, $\bar{F} = 0$; and conversely. $\mathfrak{A}$ is *everywhere dense* in a region R_m when no cell can be found within R_m which does not contain at least one point of $\mathfrak{A}$. An aggregate is said to be closed if it contains its first derivative. The aggregate $\mathfrak{A}$ may be enclosed in a finite or enumerably infinite set of cells. The volume of each such set of cells has a definite value. The minimum of this volume

is called the *upper measure* of the aggregate. The aggregate $\mathfrak{A}_1$ is said to be a *partial aggregate* of $\mathfrak{A}$ when each point of $\mathfrak{A}_1$ is in $\mathfrak{A}$. The points $\mathfrak{A}_2$ belonging to $\mathfrak{A}$ but not to $\mathfrak{A}_1$ form the aggregate *complementary* to $\mathfrak{A}_1$. To obtain the *lower measure* of the aggregate $\mathfrak{A}$ situated in the region R_m, we subtract from R_m the upper measure of the aggregate complementary to $\mathfrak{A}$. If the upper measure and lower measure are equal, the common value is called the *measure* of $\mathfrak{A}$. The upper measure of an aggregate is at least equal to its lower content, and at most equal to its upper content. For a closed aggregate the upper measure and the upper content are identical. If we have a sequence of aggregates $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$, the aggregate $\mathfrak{A}$ consisting of all points found in any one of them is called their *least common multiple*. The aggregate α containing all points common to $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is the *greatest common divisor* of that sequence. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are complementary partial aggregates of $\mathfrak{A}$, they are said to be *unmixed* if the aggregate of frontier points common to them is discrete.

In this article the restriction is put on the field of integration that the frontier be discrete. The field possessing content is denoted by A to distinguish it from the general field $\mathfrak{A}$. The function $f(x_1, \cdots, x_m)$ defined over A may take on any value unless otherwise stated. If in the field A we hold p coördinates $x_1, \cdots, x_p$ fixed, the point $(x_1, \cdots, x_m)$ ranges over an aggregate of q dimensions $(p + q = m)$ which is called $\mathfrak{C}$. If the integral of f is taken over $\mathfrak{C}$, the resulting function $F(x_1, \cdots, x_p)$ is a function of p variables. The function F is defined over an aggregate $\mathfrak{B}$ in p-dimensional space and may permit of integration. The resulting integral is written

$$\int_{\mathfrak{B}} F(x_1, \cdots, x_p) = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f(x_1, \cdots, x_m),$$

and is called the iterated integral. The simplest case is the double integral $(m = 2)$ which is broken up into two single integrations. Throughout the discussion of the reduction of a multiple integral, the aggregate $\mathfrak{B}$ is assumed to possess content or to be closed and is then denoted by B.

In the memoir we adopt the definition of a multiple integral of a *limited* function $f(x_1, \cdots, x_m)$ introduced by Professor PIERPONT,* which includes as special cases the integrals of JORDAN, STOLZ and DE LA VALLÉE-POUSSIN. Since in this definition the arrangement of the frontier points is in no manner restricted, the difficulties which ordinarily arise from a discussion of these points are entirely avoided. This definition is as follows.

Let $\mathfrak{A}$ be a limited field in m-dimensions, over which a limited function $f(x_1, \cdots, x_m)$ is defined. Effect a division of space of norm δ into cells $\delta_1, \delta_2, \cdots$ not necessarily rectangular. Take those cells which contain points of $\mathfrak{A}$. Let

* *Lectures*, p. 528.

M_i and m_i denote respectively the maximum and minimum of f in δ_i. When δ approaches zero, the two sums $\Sigma M_i \delta_i$, $\Sigma m_i \delta_i$ approach certain limits which are denoted by

$$\overline{\int_{\mathfrak{A}}} f(x_1, \cdots, x_m)\, d\mathfrak{A}, \qquad \underline{\int_{\mathfrak{A}}} f(x_1, \cdots x_m)\, d\mathfrak{A},$$

and are called the upper and lower integrals of f in $\mathfrak{A}$. If these are equal, the common value is called the integral of f in $\mathfrak{A}$ and is written

$$\int_{\mathfrak{A}} f(x_1, \cdots, x_m) = \int_{\mathfrak{A}} f.$$

Two methods for defining *improper* integrals in a limited field $\mathfrak{A}$ are available, being generalizations respectively of the definition ordinarily followed and of the definition due to DE LA VALLÉE-POUSSIN. The first method is that which is followed by Professor PIERPONT,* while the second is used by the writer in a previous article.† For convenience we insert here the definitions used in this previous article.

With the function f is associated the auxiliary function $f_{\lambda_1\lambda_2}$ defined thus:

$$f_{\lambda_1\lambda_2} = f \text{ if } -\lambda_2 \leqq f \leqq \lambda_1; \quad f_{\lambda_1\lambda_2} = \lambda_1 \text{ if } f > \lambda_1; \quad f_{\lambda_1\lambda_2} = -\lambda_2 \text{ if } f < -\lambda_2.$$

Provided the limits exist, we define the improper integrals as follows:

$$(1) \qquad \int_{\mathfrak{A}} f = \lim_{\lambda_1=\infty,\, \lambda_2=\infty} \int_{\mathfrak{A}} f_{\lambda_1\lambda_2}, \qquad (2) \quad \overline{\int_{\mathfrak{A}}} f = \lim_{\lambda_1=\infty,\, \lambda_2=\infty} \overline{\int_{\mathfrak{A}}} f_{\lambda_1\lambda_2},$$

and so forth. The integral itself is said "to exist" only if the limit is finite. In case the limit is definitely infinite (either $+\infty$ or $-\infty$), this will be expressed by saying that the integral exists infinitely. If the limit does not exist, the function is not integrable. In the iterated integral

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$$

the integrand

$$\int_{\mathfrak{C}} f$$

may be infinite, and we introduce the notation:

*Transactions of the American Mathematical Society, vol. 7 (1906), p. 158.

†Transactions, vol. 7 (1906), p. 449. It may be noted that the discussion given in the present article will hold if in place of f_λ as here defined we read f_n where the latter symbol is that defined by DE LA VALLÉE-POUSSIN (*Cours d'Analyse*, vol. 2, p. 89) and used by him in the exposition of the reduction of the double integral.

$$\left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2} = \int_{\mathfrak{C}} f \quad \text{if } -\mu_2 \leqq \int_{\mathfrak{C}} f \leqq \mu_1,$$

$$\left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2} = \mu_1 \quad \text{if } \int_{\mathfrak{C}} f > \mu_1,$$

$$\left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2} = -\mu_2 \quad \text{if } \int_{\mathfrak{C}} f < -\mu_2.$$

If when μ_1, μ_2 approach infinity in any manner a unique limit is obtained, we take by definition

$$\underline{\int}_{\mathfrak{B}} \int_{\mathfrak{C}} f = \lim_{\mu_1=\infty,\ \mu_2=\infty} \underline{\int}_{\mathfrak{B}} \left[\int_{\mathfrak{C}} f\right]_{\mu_1\mu_2},$$

and so forth.

Whenever

$$\int_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f = \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f, \tag{1}$$

the integral

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f = \int_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f$$

is said to exist although for some points of $\mathfrak{B}$ the integrand

$$\int_{\mathfrak{C}} f$$

is not determinate. If we denote by $\mathfrak{B}_\sigma$ the partial aggregate of $\mathfrak{B}$ where

$$\bar{\int}_{\mathfrak{C}} f - \underline{\int}_{\mathfrak{C}} f > \sigma,$$

it is evident that if $\sigma > 0$, $\bar{\mathfrak{B}}_\sigma = 0$. Throughout this article the notation

$$\int_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f$$

is used implying the existence and equality of the integrals of (1).

The writer has shown * that the improper integrals defined by the two methods are absolutely convergent. One advantage of the second method is that it may be shown (theorem 4) that in order that the integral may exist, the points of infinite discontinuity of f must be discrete; if the first method is used, this must be expressly assumed. Another advantage of this method is the pos-

*Transactions, loc. cit., theorems 6 and 12. For a correction of the proof of theorem 6 see the note on page 371 of the present memoir.

sibility of integrating along a section for every point of which the function is infinite, the other definition not permitting of this operation. Apart from these differences, as I have shown,* the existence of one of the improper integrals defined by these methods is a necessary and sufficient condition for the existence of the other. The *lower* integral as defined by the second method may exist for a positive function when it has points of infinite discontinuity everywhere dense; while no lower integral is possible by the first method.

The various properties of these integrals have already been developed.† We mention a few that are characteristic.

I. If f is integrable in $\mathfrak{A}$, it is integrable in any partial field $\mathfrak{A}_1$.

II. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed complementary partial aggregates of $\mathfrak{A}$ in which f is integrable, then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f.$$

III. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are complementary partial aggregates of $\mathfrak{A}$, over which a function $f \geqq 0$ is defined, then

$$\int_{\mathfrak{A}} f \leqq \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f,$$

provided the integrals exist.

IV. If the integral over $\mathfrak{A}$ exists, then for every $\epsilon > 0$ there is a $\delta > 0$ such that for any set of unmixed partial aggregates $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ of which the total upper content is less than δ the sum of the integrals over these partial fields is less in absolute value than ϵ.

V. It is possible to change the value of the integrand at a discrete aggregate of points without altering the value of the integral. The new values may be finite or infinite.

VI. If f and g are integrable in $\mathfrak{A}$, the integral of the sum is equal to the sum of the integrals.

It may be noted that the theorems of this paper hold in case the integrals exist when the definition of HARNACK or any of its extensions is used. If $\mathfrak{C}$ is a one-dimensional aggregate, the integral of f over $\mathfrak{C}$ may exist only non-absolutely. In this case the ordinary (HARNACK) definition is more general than that of DE LA VALLÉE-POUSSIN. Although the notation used throughout this memoir is that of the narrower integral, the discussion will hold for the more general case.

* Loc. cit., theorems 16 and 17. This result was given to the Society December 28, 1905. HOBSON has proved this for a less general case in the Proceedings of the London Mathematical Society, ser. 2, vol. 4 (June, 1906), p. 139.

† Loc. cit.

§ 2. *Theorems on aggregates and integrals.*

THEOREM 1. *Let $\mathfrak{A}$ be a limited point aggregate in m-dimensional space, and $\mathfrak{B}$ the p-dimensional ($p < m$) aggregate which is the projection of $\mathfrak{A}$ on a p-dimensional section. Denote by $\mathfrak{B}_1$ a discrete partial aggregate of $\mathfrak{B}$ and by $\mathfrak{A}_1$ the partial aggregate of $\mathfrak{A}$ which has for its projection $\mathfrak{B}_1$. Then $\mathfrak{A}_1$ is discrete.*

Enclose the points of $\mathfrak{B}_1$ in p-dimensional cubes of volume small at pleasure. Complete the m-dimensional parallelopipeds enclosing $\mathfrak{A}_1$. The volume of these can evidently be made as small as is desired.

THEOREM 2. *Let $A_1 \leqq A_2 \leqq A_3 \leqq \cdots$ be a series of partial aggregates of a limited point aggregate $\mathfrak{A}$ and let $\lim_{n=\infty} \bar{A}_n = \eta$. Denote by L the least common multiple of the series and by α the partial aggregate of $\mathfrak{A}$ complementary to L. Then if $\mathfrak{A}$ is closed or possesses content, $\bar{\mathfrak{A}} \geqq \bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$.*

Since α is a partial aggregate of $\mathfrak{A}$ it is evident that $\bar{\mathfrak{A}} \geqq \bar{\alpha}$. Denote by σ_n the upper content of A_n. Then $\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \eta$. By definition it is possible to enclose the points of A_1 in cells of volume less than $\sigma_1 + \frac{1}{2}\epsilon$ and the points of A_n, not already enclosed, in cells of volume less than $\sigma_n - \sigma_{n-1} + \epsilon/2^n$. The sequence $A_1, A_2, \cdots$ may then be enclosed in an enumerable set of cells of volume less than $\eta + \epsilon$. Since the points of α may be enclosed in cells of volume less than $\bar{\alpha} + \epsilon$, it follows that the points of $\mathfrak{A}$ may be enclosed in a finite or enumerably infinite set of cells in volume less than $\eta + \bar{\alpha} + 2\epsilon$.

If $\mathfrak{A}$ possess content, it is evident that $\mathfrak{A} = \underline{\mathfrak{A}} < \eta + \bar{\alpha} + 2\epsilon$, since in calculating the lower content only those cells are taken which contain no points except those of $\mathfrak{A}$. This being true for all values of ϵ, we have $\bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$.

If $\mathfrak{A}$ is closed, upper measure $\mathfrak{A} = \bar{\mathfrak{A}}$. Since $\bar{\mathfrak{A}}$ is the minimum volume of cells enclosing $\mathfrak{A}$, it follows that $\eta + \bar{\alpha} + 2\epsilon \geqq \bar{\mathfrak{A}}$, or $\bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta - 2\epsilon$. Passing to the limit we have $\bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$.

COROLLARY. *Let $\mathfrak{A}_1 \leqq \mathfrak{A}_2 \leqq \cdots$ be a sequence of aggregates whose least common multiple is $\mathfrak{A}$. If $\mathfrak{A}$ possesses content or is closed, $\lim_{n=\infty} \bar{\mathfrak{A}}_n = \bar{\mathfrak{A}}$.*

In this case α possesses no points. Hence $0 = \bar{\alpha} \geqq \bar{\mathfrak{A}} - \eta$, or $\bar{\mathfrak{A}} \leqq \eta$. Since however $\bar{\mathfrak{A}} \geqq \eta$, it follows that $\lim_{n=\infty} \bar{\mathfrak{A}}_n = \eta = \bar{\mathfrak{A}}$.

THEOREM 3. *Let $\mathfrak{A}$ be an aggregate discrete in m dimensions, and $\mathfrak{B}$ and $\mathfrak{C}$ the corresponding* aggregates in p and q dimensions. Let $\mathfrak{B}_1$ denote the partial aggregate of $\mathfrak{B}$ for which $\bar{\mathfrak{C}} = 0$, and $\mathfrak{B}_2$ the complementary aggregate of $\mathfrak{B}$. Then measure $\mathfrak{B}_2 = 0$, and if $\mathfrak{B}$ possesses content, $\bar{\mathfrak{B}} = \bar{\mathfrak{B}}_1$.*

Choose a sequence $\sigma_1 > \sigma_2 > \sigma_3 > \cdots$ with the limit 0 and denote by $\mathfrak{B}_{\sigma_i}$ the partial aggregate of $\mathfrak{B}$ where $\bar{\mathfrak{C}} \geqq \sigma_i$. It has been shown by SCHÖNFLIES† that $\mathfrak{B}_{\sigma_i}$ is discrete. Since the measure of each of the enumerable sequence of aggregates $\mathfrak{B}_{\sigma_1}$, $\mathfrak{B}_{\sigma_2} - \mathfrak{B}_{\sigma_1}$, $\mathfrak{B}_{\sigma_3} - \mathfrak{B}_{\sigma_2}$, $\cdots$ is zero, it follows by a familiar

* See p. 342 for definition.

† *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* (1900), p. 96.

theorem that measure $\mathfrak{B}_2 = 0$. Since now $\mathfrak{B}_2$ is the least common multiple of $\mathfrak{B}_{\sigma_1}, \mathfrak{B}_{\sigma_2}, \cdots$ and since $\lim_{n=\infty} \mathfrak{B}_{\sigma_n} = 0$, it follows from theorem 2 that if $\mathfrak{B}$ possesses content, $\bar{\mathfrak{B}}_1 = \mathfrak{B}$.

COROLLARY. *Let $\mathfrak{A}$ be an aggregate which possesses content. Denote by $\mathfrak{B}_2$ the points of $\mathfrak{B}$ when the section $\mathfrak{C}$ does not possess content and by $\mathfrak{B}_1$ the complementary partial aggregate of $\mathfrak{B}$. Then measure $\mathfrak{B}_2 = 0$, and if $\mathfrak{B}$ possesses content, $\bar{\mathfrak{B}} = \bar{\mathfrak{B}}_1$.*

Since the frontier points of $\mathfrak{A}$ form a discrete aggregate, this is an immediate consequence of the theorem.

THEOREM 4. *Let the function f be defined over the limited aggregate $\mathfrak{A}$ and be integrable in $\mathfrak{A}$. If $\mathfrak{A}_\lambda$ denotes the partial aggregate where $|f| > \lambda$, it is possible to take λ so large that $\bar{\mathfrak{A}}_\lambda$ is small at pleasure.**

Since the integral converges it converges absolutely, as was shown in the former paper.† Let

$$\int_{\mathfrak{A}} |f| = M.$$

Choose ϵ small at pleasure and take $\lambda > M/\epsilon$. Then

$$M = \int_{\mathfrak{A}} |f| \geqq \int_{\mathfrak{A}_\lambda} |f| \geqq \int_{\mathfrak{A}} \lambda \geqq \lambda \bar{\mathfrak{A}}_\lambda.$$

Therefore

$$\bar{\mathfrak{A}}_\lambda \leqq \frac{M}{\lambda} < \epsilon.$$

The following theorem is obvious.

THEOREM 5. *Let $f \geqq 0$ be a limited function defined over a limited aggregate $\mathfrak{A}$. Adjoin to $\mathfrak{A}$ any aggregate of points and call the resulting aggregate R. Define the new function g as follows:*

$$g = f \text{ for points of } \mathfrak{A}, \qquad g = 0 \text{ for other points of } R.$$

Then

$$\bar{\int}_R g = \bar{\int}_{\mathfrak{A}} f.$$

THEOREM 6. *Let $\mathfrak{A}$ be a limited aggregate, not necessarily possessing content, over which a limited function $f \geqq 0$ is defined. Then*

$$\bar{\int}_{\mathfrak{A}} f \geqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f. \tag{1}$$

* This theorem is true for a function of more than one variable. It is true for a function of one variable that is absolutely integrable, in other words for a function which is integrable by the method of DE LA VALLÉE-POUSSIN.

† Theorem 12, Transactions, vol. 7 (1906), p. 455.

Enclose the aggregate $\mathfrak{A}$ in a cube the totality of whose points is denoted by R. Let T and S be the aggregates of R corresponding to $\mathfrak{C}$ and $\mathfrak{B}$. Define the auxiliary function g as follows:

$$g = f \text{ for points of } \mathfrak{A}, \quad g = 0 \text{ for other points of } R.$$

Now R possesses content, and by Professor PIERPONT's *Lectures*, § 733,

$$\bar{\int}_R g \geqq \bar{\int}_S \bar{\int}_T g,$$

and this by theorem 5 gives relation (1).

THEOREM 7. *Let the functions* $f(x_1, \cdots, x_m) \geqq 0$, $g(x_1, \cdots, x_m) \geqq 0$ *be defined over a limited aggregate* $\mathfrak{A}$. *Then*

$$(1) \quad \bar{\int}_{\mathfrak{A}} f + \bar{\int}_{\mathfrak{A}} g \geqq \bar{\int}_{\mathfrak{A}} (f+g) \geqq \bar{\int}_{\mathfrak{A}} f + \underline{\int}_{\mathfrak{A}} g \geqq \underline{\int}_{\mathfrak{A}} (f+g) \geqq \underline{\int}_{\mathfrak{A}} f + \underline{\int}_{\mathfrak{A}} g,$$

and provided the upper and lower integrals of $f-g$ *exist finitely or infinitely and the inequalities have a sense,*

$$(2) \quad \bar{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g \geqq \bar{\int}_{\mathfrak{A}} (f-g) \geqq \underline{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g \geqq \underline{\int}_{\mathfrak{A}} (f-g) \geqq \underline{\int}_{\mathfrak{A}} f - \bar{\int}_{\mathfrak{A}} g.$$

If f and g are limited functions, for every cell of $\mathfrak{A}$ we have the following inequalities:

$$\max f + \max g \geqq \max (f+g) \geqq \max f + \min g \geqq \min (f+g) \geqq \min f + \min g,$$

$$\max f - \min g \geqq \max (f-g) \geqq \min f - \min g \geqq \min (f-g) \geqq \min f - \max g.$$

Relations (1) and (2) follow immediately from these inequalities. When f and g are unlimited, we note that

$$f_{2\lambda} + g_{2\lambda} \geqq (f+g)_{2\lambda} \geqq f_\lambda + g_\lambda.$$

Hence from the previous proof for f and g limited,

$$\bar{\int}_{\mathfrak{A}} f_{2\lambda} + \bar{\int}_{\mathfrak{A}} g_{2\lambda} \geqq \bar{\int}_{\mathfrak{A}} (f_{2\lambda} + g_{2\lambda}) \geqq \bar{\int}_{\mathfrak{A}} (f+g)_{2\lambda} \geqq \bar{\int}_{\mathfrak{A}} (f_\lambda + g_\lambda) \geqq \bar{\int}_{\mathfrak{A}} f_\lambda + \underline{\int}_{\mathfrak{A}} g_\lambda.$$

On passing to the limit for $\lambda = \infty$, we obtain

$$\bar{\int}_{\mathfrak{A}} f + \bar{\int}_{\mathfrak{A}} g \geqq \bar{\int}_{\mathfrak{A}} (f+g) \geqq \bar{\int}_{\mathfrak{A}} f + \underline{\int}_{\mathfrak{A}} g.$$

In a similar manner the other inequalities of (1) are proved.

To establish (2), let us first assume that f is unlimited and g limited. Then for $\lambda > g$ we have the inequality

$$f_{2\lambda} - g \geqq (f-g)_\lambda \geqq f_\lambda - g.$$

Hence from the previous proof for f and g limited,

$$\bar{\int}_{\mathfrak{A}} f_{2\lambda} - \underline{\int}_{\mathfrak{A}} g \geqq \bar{\int}_{\mathfrak{A}} (f_{2\lambda} - g) \geqq \bar{\int}_{\mathfrak{A}} (f-g)_\lambda \geqq \bar{\int}_{\mathfrak{A}} (f_\lambda - g) \geqq \underline{\int}_{\mathfrak{A}} f_\lambda - \underline{\int}_{\mathfrak{A}} g.$$

On passing to the limit for $\lambda = \infty$ we have

$$\bar{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g \geqq \bar{\int}_{\mathfrak{A}} (f-g) \geqq \underline{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g,$$

and the remaining inequalities of (2) are similarly established.

When f and g are unlimited, it may be noted that for all values of λ

$$\bar{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g_\lambda \geqq \bar{\int}_{\mathfrak{A}} (f - g_\lambda) \geqq \bar{\int}_{\mathfrak{A}} (f-g).$$

A passage to the limit for $\lambda = \infty$ gives the first inequality of (2). If the second inequality of (2) does not hold, λ can be found so large that

$$\bar{\int}_{\mathfrak{A}} (f-g) < \underline{\int}_{\mathfrak{A}} f_\lambda - \underline{\int}_{\mathfrak{A}} g,$$

or, by taking negatives,

$$\underline{\int}_{\mathfrak{A}} g - \underline{\int}_{\mathfrak{A}} f_\lambda < \underline{\int}_{\mathfrak{A}} (g-f) \leqq \underline{\int}_{\mathfrak{A}} (g - f_\lambda).$$

Here g is unlimited and f_λ limited, and this relation contradicts the third inequality of (2) which has been proved for such a case. Hence the second inequality of (2) holds for all cases. The remaining inequalities may be established by taking the negative of the first two.

COROLLARY 1. *The relations* (1) *and* (2) *hold if* $f \geqq -G$, $g \geqq -G$, *where* G *is arbitrarily large but fixed.*

For

$$\underline{\int}_{\mathfrak{A}} (f-g) = \underline{\int}_{\mathfrak{A}} [(f+G) - (g+G)] \geqq \underline{\int}_{\mathfrak{A}} (f+G) - \bar{\int}_{\mathfrak{A}} (g+G)$$

$$= \underline{\int}_{\mathfrak{A}} f + \underline{\int}_{\mathfrak{A}} G - \bar{\int}_{\mathfrak{A}} g - \underline{\int}_{\mathfrak{A}} G = \underline{\int}_{\mathfrak{A}} f - \bar{\int}_{\mathfrak{A}} g,$$

which gives the last relation of (2). The other relations are established in a similar manner.

COROLLARY 2. *If g is integrable,*

$$\bar{\int}_{\mathfrak{A}}(f+g)=\bar{\int}_{\mathfrak{A}}f+\int_{\mathfrak{A}}g, \qquad \underline{\int}_{\mathfrak{A}}(f+g)=\underline{\int}_{\mathfrak{A}}f+\int_{\mathfrak{A}}g,$$

$$\bar{\int}_{\mathfrak{A}}(f-g)=\bar{\int}_{\mathfrak{A}}f-\int_{\mathfrak{A}}g, \qquad \underline{\int}_{\mathfrak{A}}(f-g)=\underline{\int}_{\mathfrak{A}}f-\int_{\mathfrak{A}}g.$$

THEOREM 8. *Suppose the upper or lower integral of $f(x_1, \cdots, x_m)$ over $\mathfrak{A}$ to exist. Then the upper and lower integral respectively of $f-f_{\lambda_1\lambda_2}$ exists, and correspondingly*

$$(1) \qquad \bar{\int}_{\mathfrak{A}}f-\bar{\int}_{\mathfrak{A}}f_{\lambda_1\lambda_2}=\bar{\int}_{\mathfrak{A}}(f-f_{\lambda_1\lambda_2}), \qquad \underline{\int}_{\mathfrak{A}}f-\underline{\int}_{\mathfrak{A}}f_{\lambda_1\lambda_2}=\underline{\int}_{\mathfrak{A}}(f-f_{\lambda_1\lambda_2}).$$

For a limited function f this is true, since for every cell,

$$\max f-\max f_{\lambda_1\lambda_2}=\max(f-f_{\lambda_1\lambda_2}), \qquad \min f-\min f_{\lambda_1\lambda_2}=\min(f-f_{\lambda_1\lambda_2}).$$

If f is unlimited, the following relation holds for all points of $\mathfrak{A}$:

$$f_{\lambda_1+\lambda,\,\lambda_2+\lambda}-f_{\lambda_1\lambda_2}=(f-f_{\lambda_1\lambda_2})_{\lambda,\,\lambda}.$$

By means of this equality, it is evident that

$$\bar{\int}_{\mathfrak{A}}f_{\lambda_1+\lambda,\,\lambda_2+\lambda}-\bar{\int}_{\mathfrak{A}}f_{\lambda_1\lambda_2}=\bar{\int}_{\mathfrak{A}}(f-f_{\lambda_1\lambda_2})_{\lambda,\,\lambda}; \qquad \underline{\int}_{\mathfrak{A}}f_{\lambda_1+\lambda,\,\lambda_2+\lambda}-\underline{\int}_{\mathfrak{A}}f_{\lambda_1\lambda_2}=\underline{\int}_{\mathfrak{A}}(f-f_{\lambda_1\lambda_2})_{\lambda,\,\lambda}.$$

By passing to the limit for $\lambda=\infty$ the theorem is established.

It may be noted that if one of the integrals exists infinitely, the theorem is still valid.

THEOREM 9. *Let $f(x_1, \cdots, x_m)$ be defined over $\mathfrak{A}$. Define the auxiliary functions $g(x_1, \cdots, x_m)$, $h(x_1, \cdots, x_m)$ as follows:*

$g=f$ *for* $f>0$, $g=0$ *for* $f\leqq 0$; $\qquad h=-f$ *for* $f<0$, $h=0$ *for* $f\geqq 0$.

Then

$$(1) \qquad \int_{\mathfrak{A}}f=\int_{\mathfrak{A}}g-\underline{\int}_{\mathfrak{A}}h,$$

$$(2) \qquad \underline{\int}_{\mathfrak{A}}f=\underline{\int}_{\mathfrak{A}}g-\bar{\int}_{\mathfrak{A}}h,$$

provided in each equation one of the integrals on the right exists finitely.

If first f is assumed to be a limited function, it will be observed that the cells of a division may be separated into two classes; 1) those in which $g = 0$ throughout the cell, and 2) those in which $f > 0$ for at least one point of the cell. In either case $\max(g - h) = \max g - \min h$. By the usual process this gives (1).

If now f is unlimited, we have for all values of λ_1, λ_2

$$\overline{\int_{\mathfrak{A}}} f_{\lambda_1\lambda_2} = \overline{\int_{\mathfrak{A}}} (g - h)_{\lambda_1\lambda_2} = \overline{\int_{\mathfrak{A}}} g_{\lambda_1} - \underline{\int_{\mathfrak{A}}} h_{\lambda_2},$$

whence equation (1) at once follows by passing to the limit for $\lambda_1 = \infty$, $\lambda_2 = \infty$. Relation (2) is established in a similar manner.

THEOREM 10. *Let $f(x_1, \cdots, x_m)$ be defined over a limited aggregate $\mathfrak{A}$. Let $\mathfrak{B}$ and $\mathfrak{C}$ be the corresponding aggregates in p and q dimensions. If the integrals*

$$\int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} \overline{f}, \qquad \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} \overline{f}_{\lambda_1\lambda_2}$$

exist, then

$$(1) \qquad \left| \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} \overline{(f - f_{\lambda_1\lambda_2})} \right| \leqq \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} |f - f_{\lambda_1\lambda_2}| \leqq \underline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}}} |f - f_{\lambda_1\lambda_2}|.$$

For consider the upper and lower integrals of f over the sections $\mathfrak{C}$. For various points of $\mathfrak{B}$ the following cases arise: (i) At least one of these integrals does not exist either finitely or infinitely; (ii) one integral (or both) exists infinitely; (iii) the upper integral exceeds the lower integral by an amount equal to or greater than a given $\eta > 0$; (iv) the difference between the upper and lower integrals is less than η. The points of $\mathfrak{B}$ belonging to cases (i) and (ii), and to (iii) for a given value of η, form a discrete aggregate. Denoting by $\mathfrak{B}_\sigma$ the points of $\mathfrak{B}$ remaining when this discrete set is enclosed in cells of volume less than σ, it follows from property (iv) of integrals (p. 345) that for a sufficiently small σ

$$\left| \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} \overline{f} - \int_{\mathfrak{B}_\sigma} \underline{\int_{\mathfrak{C}}} \overline{f} \right| < \epsilon.$$

In a similar manner, we have for a properly chosen σ

$$(2) \quad \left| \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} \overline{f}_{\lambda_1\lambda_2} - \int_{\mathfrak{B}_\sigma} \underline{\int_{\mathfrak{C}}} \overline{f}_{\lambda_1\lambda_2} \right| < \epsilon, \quad \left| \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}} \overline{(f - f_{\lambda_1\lambda_2})} - \int_{\mathfrak{B}_\sigma} \underline{\int_{\mathfrak{C}}} \overline{(f - f_{\lambda_1\lambda_2})} \right| < \epsilon.$$

Since in any cell of $\mathfrak{C}$ the oscillation of the function $|f - f_{\lambda_1\lambda_2}| = |f| - |f_{\lambda_1\lambda}$ is equal to or less than the oscillation of $f - f_{\lambda_1\lambda_2}$, it follows that

$$\bar{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| - \underline{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| \leqq \bar{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}) - \underline{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}).$$

By means of theorem 8 this becomes for every point of $\mathfrak{B}_\sigma$

$$0 \leqq \bar{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| - \underline{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| \leqq \bar{\int}_{\mathfrak{C}} f - \underline{\int}_{\mathfrak{C}} f - \left(\bar{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} - \underline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right) < \eta,$$

whence we have with the aid of the second inequality of (2), theorem 7,

$$0 \leqq \underline{\int}_{\mathfrak{B}_\sigma} \bar{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| - \underline{\int}_{\mathfrak{B}_\sigma} \underline{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| \leqq \eta \bar{\mathfrak{B}}_\sigma. \tag{3}$$

Since now

$$\left| \bar{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}) \right| \leqq \bar{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}|,$$

and since an integrable function is absolutely integrable, we have

$$\int_{\mathfrak{B}_\sigma} \left| \bar{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}) \right| \leqq \underline{\int}_{\mathfrak{B}_\sigma} \bar{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}|.$$

And *a fortiori*

$$\left| \int_{\mathfrak{B}_\sigma} \bar{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}) \right| \leqq \underline{\int}_{\mathfrak{B}_\sigma} \bar{\int}_{\mathfrak{C}} |f- {}_{\lambda_1\lambda_2}|. \tag{4}$$

By combining (2) and (3) with (4) we obtain

$$\left| \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}) \right| - \epsilon \leqq \underline{\int}_{\mathfrak{B}_\sigma} \underline{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| + \eta \bar{\mathfrak{B}}_\sigma \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| + \eta \bar{\mathfrak{B}}_\sigma.$$

Since ϵ and η are small at pleasure, this gives

$$\left| \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} (f-f_{\lambda_1\lambda_2}) \right| \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}| \leqq \underline{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} |f-f_{\lambda_1\lambda_2}|.$$

THEOREM 11. *Let the limited function $f \geqq 0$ be defined over a limited aggregate $\mathfrak{A}$. Let $A_1 \leqq A_2 \leqq \cdots$ be a series of partial aggregates of $\mathfrak{A}$ such that $\lim_{n=\infty} \bar{A}_n = \bar{\mathfrak{A}}$. At every point of any A_n let $f < \sigma$. Then*

$$\underline{\int}_{\mathfrak{A}} f \leqq \sigma \bar{\mathfrak{A}}.$$

Denote by L the least common multiple of $A_1, A_2, \cdots$. Then since L is a partial aggregate of $\bar{\mathfrak{A}}$, $\bar{A}_n \leqq \bar{L} \leqq \bar{\mathfrak{A}}$. This is true for all values of n, and passing to the limit we have $\bar{L} = \bar{\mathfrak{A}}$. For ϵ small at pleasure it is possible to

effect a division of space such that the volume of cells containing at least one point of $\mathfrak{A}$ is less than $\bar{\mathfrak{A}} + \epsilon$. Since the minimum volume of cells containing at least one point of L is $\bar{\mathfrak{A}}$, the volume of cells containing a point of $\mathfrak{A}$ and not containing one of L must be less than ϵ. Now f is limited, and the integral over the points in these cells is small at pleasure. In the remaining cells the minimum of f is certainly not greater than σ. Hence

$$\underline{\int}_{\mathfrak{A}} f \leqq \sigma \bar{\mathfrak{A}}.$$

COROLLARY. *Denote by f_n a limited function which in the points of A_n is equal to f and which at the remaining points of $\mathfrak{A}$ is arbitrary except that it be positive and limited. Then for η small at pleasure, n may be taken so great that*

$$\underline{\int}_{\mathfrak{A}} f_n < \sigma \bar{\mathfrak{A}} + \eta. \tag{1}$$

For n may be so chosen that $\bar{A}_n > \bar{\mathfrak{A}} - \epsilon$. In other words, the volume of the cells where the minimum of f_n is greater than σ, while the minimum of f is equal to or less than σ, is small at pleasure. By reasoning analogous to that in theorem 11 we arrive at relation (1).

THEOREM 12. *Let $f(x_1, \cdots, x_m)$ be limited in the field A which possesses content. Then*

$$\underline{\int}_A f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \bar{\int}_A f, \qquad \underline{\int}_A f \leqq \underline{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f \leqq \bar{\int}_A f.$$

The proof is essentially that given by Professor PIERPONT* for a slightly less general case.

COROLLARY. *If the multiple integral exists, then*

$$\int_A f = \int_{\mathfrak{B}} \bar{\int}_{\mathfrak{C}} f = \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

§ 3. *Integration of a sequence of functions.*

Let us consider the integration of a sequence of functions $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m), \cdots$ *defined over a limited aggregate A which either possesses content or is closed.* We assume that for each point of A the sequence f_1, $f_2, \cdots$ has a limiting value which is finite or definitely infinite. The aggregate of such limiting values defines a new function

* *Lectures*, § 733.

$$f(x_1, \cdots, x_m) = \lim_{n=\infty} f_n(x_1, \cdots, x_m).$$

Unless otherwise stated, no restriction is put on the continuity of the functions $f, f_1, f_2, \cdots$, which may take on finite or infinite values.

THEOREM 13. *Let $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m), \cdots$ be for each $(x_1, \cdots, x_m)$ in A a monotone sequence of functions converging to $f(x_1, \cdots, x_m)$.*

1) *If the functions are positive and the sequence is increasing,*

$$\int_A^{\bar{}} f \geqq \lim_{n=\infty} \int_A^{\bar{}} f_n \geqq \int_{\underline{A}} f \geqq \lim_{n=\infty} \int_{\underline{A}} f_n. \tag{1}$$

2) *If the functions are negative* and the sequence is decreasing,*

$$\lim_{n=\infty} \int_A^{\bar{}} f_n \geqq \int_A^{\bar{}} f \geqq \lim_{n=\infty} \int_{\underline{A}} f_n \geqq \int_{\underline{A}} f. \tag{2}$$

To prove (1) let us assume in the first place that f is limited throughout A. Choose $\sigma > 0$ small at pleasure, and denote by A_n the partial aggregate of A where $f - f_n \leqq \sigma$. It is evident that $A_n \leqq A_{n+1}$. The least common multiple of the sequence $A_1, A_2, \cdots$ is the aggregate A. For, since $f_n(P)$ converges to $f(P)$, it follows that any point P of A belongs to the sequence $A_\nu, A_{\nu+1}, \cdots$ for a sufficiently large ν. Hence by the corollary of theorem 2, $\lim_{n=\infty} \bar{A}_n = \bar{A}$. Applying now the corollary to theorem 11, we have by proper choice of n, η, and σ,

$$\int_{\underline{A}} (f - f_n) < \sigma \bar{A} + \eta < \epsilon.$$

Hence from theorem 7

$$\int_{\underline{A}} f - \int_A^{\bar{}} f_n < \epsilon,$$

where the expression on the left is not necessarily positive or zero. By passing to the limit the second relation of (1) is established.

If now f is unlimited, two cases arise according as the lower integral of f is finite or infinite. If finite, then for proper choice of λ

$$\lim_{n=\infty} \int_A^{\bar{}} f_n \geqq \lim_{n=\infty} \int_A^{\bar{}} (f_n)_\lambda \geqq \int_{\underline{A}} f_\lambda \geqq \int_{\underline{A}} f - \epsilon.$$

Since this is true for infinitesimal values of ϵ, we have the desired relation. If, on the other hand, the lower integral of f is infinite, λ may be so chosen that for M great at pleasure,

* It will hereafter be assumed without express statement that if a theorem is true for $f \geqq 0$, a corresponding theorem is true for $f \leqq 0$.

$$\underline{\int}_A f_\lambda > M.$$

Hence

$$\lim_{n=\infty} \bar{\int}_A f_n \geqq \lim_{n=\infty} \bar{\int}_A (f_n)_\lambda \geqq \underline{\int}_A f_\lambda > M,$$

and

$$\lim_{n=\infty} \bar{\int}_A f_n = \infty.$$

This establishes the second relation of (1) for all cases. The other parts are obvious since $f \geqq f_n$. To prove (2) it suffices to take the negative of the terms of (1).

COROLLARY. *If the positive functions $f_1, f_2, \cdots$ are integrable in A, then*

$$\lim_{n=\infty} \int_A f_n = \int_A f.$$

It may be remarked that the relations (1) of this theorem and corollary are true if each function of the sequence is greater than $-G$, where G is arbitrarily large but fixed. (See theorem 7, corollary 1.)

The points where $f - f_n \geqq \sigma > 0$ may be everywhere dense for all values of n. The complementary aggregate, however, is such that its upper content may, by proper choice of n, be made to differ by an infinitesimal from the upper content of A. Examples may be given where

$$\lim_{n=\infty} \bar{\int}_A f_n = 0, \qquad \bar{\int}_A f = +M,$$

and also where

$$\lim_{n=\infty} \underline{\int}_A f_n = 0, \qquad \underline{\int}_A f = +M,$$

M being finite or definitely infinite.

THEOREM 14. *Let $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ be for each $(x_1, \cdots, x_m)$ of A a monotone decreasing sequence of functions converging to $f(x_1, \cdots, x_m) \geqq 0$. If the upper integral of f_1 over A exists, then*

$$(1) \qquad \lim_{n=\infty} \bar{\int}_A f_n \geqq \bar{\int}_A f \geqq \lim_{n=\infty} \underline{\int}_A f_n \geqq \underline{\int}_A f.$$

We need only to establish the middle inequality.

If f_1 is limited, it is readily shown as in theorem 13 that n may be chosen such that for $\epsilon > 0$,

$$\underline{\int}_A (f_n - f) < \epsilon.$$

Hence from theorem 7

$$\underline{\int}_A f_n - \bar{\int}_A f < \epsilon.$$

This establishes the desired inequality.

If f_1 is unlimited, λ_0 may be found such that for $\lambda > \lambda_0$ (theorem 8)

$$\bar{\int}_A [f_1 - (f_1)_\lambda] = \bar{\int}_A f_1 - \bar{\int}_A (f_1)_\lambda < \epsilon.$$

And since $f_n - (f_n)_\lambda \leqq f_1 - (f_1)_\lambda$ for all values of n, we have

$$\bar{\int}_A [f_n - (f_n)_\lambda] < \epsilon. \tag{2}$$

From the preceding part of the proof n may be chosen such that

$$\bar{\int}_A [(f_n)_\lambda - f_\lambda] < \epsilon. \tag{3}$$

Hence from theorem 7 and relations (2) and (3) it follows that

$$\underline{\int}_A f_n - \bar{\int}_A f \leqq \underline{\int}_A f_n - \bar{\int}_A f_\lambda \leqq \underline{\int}_A (f_n - f_\lambda) \leqq \bar{\int}_A [f_n - (f_n)_\lambda] + \underline{\int}_A [(f_n)_\lambda - f_\lambda] < 2\epsilon,$$

which establishes (1).

COROLLARY. *If $f_1, f_2, \cdots$ are integrable in A, then*

$$\lim_{n=\infty} \int_A f_n = \bar{\int}_A f.$$

THEOREM 15. *Let $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ be a monotone sequence of integrable functions which increases simultaneously or decreases simultaneously at every $(x_1, \cdots, x_m)$ of A and converges to $f(x_1, \cdots, x_m)$. If f is integrable,*

$$\lim_{n=\infty} \int_A f_n = \int_A f.$$

Define the auxiliary functions g_n, g, h_n, h as follows:

$g_n = f_n$ for $f_n > 0$; $g_n = 0$ for $f_n \leqq 0$; $h_n = -f_n$ for $f_n < 0$; $h_n = 0$ for $f_n \geqq 0$;

$g = f$ for $f > 0$; $g = 0$ for $f \leqq 0$; $h = -f$ for $f < 0$; $h = 0$ for $f \geqq 0$.

Then it follows from the corollaries to theorems 13 and 14, that

$$\lim_{n=\infty} \int_A h_n = \int_A h, \qquad \lim_{n=\infty} \int_A g_n = \int_A g.$$

Hence

$$\lim_{n=\infty} \int_A f_n = \lim_{n=\infty} \int_A g_n - \lim_{n=\infty} \int_A h_n = \int_A g - \int_A h = \int_A f.$$

THEOREM 16. *In A let the positive functions $f_1(x_1, \cdots, x_m), f_2(x_1, \cdots, x_m)$, $\cdots$ converge to $f(x_1, \cdots, x_m)$. At the points of a partial aggregate A_1 let the sequence be monotone and increasing and at the points of the complementary aggregate A_2 let the sequence be monotone and decreasing. If the upper integral of f_1 exists, then*

$$(1) \qquad \lim_{n=\infty} \overline{\int_A} f_n \geqq \underline{\int_A} f, \qquad (2) \quad \lim_{n=\infty} \underline{\int_A} f_n \leqq \int_A f.$$

Let the auxiliary functions ϕ, ϕ_n, ψ, ψ_n be defined as follows:

$\phi_n = 0$ for points of A_2, $\quad \phi_n = f_n$ for points of A_1, $\quad \lim_{n=\infty} \phi_n = \phi$.

$\psi_n = f_n$ for points of A_2, $\quad \psi_n = 0$ for points of A_1, $\quad \lim_{n=\infty} \psi_n = \psi$.

Then $f = \phi + \psi$, $f_n = \phi_n + \psi_n$. If the lower integral of ϕ is limited, it follows from the proof of theorem 13 that for $\epsilon > 0$, n may be found such that

$$\underline{\int_A} (\phi - \phi_n) \leqq \epsilon.$$

Since $f - f_n = \phi - \phi_n + \psi - \psi_n \leqq \phi - \phi_n$, this gives

$$\underline{\int_A} (f - f_n) \leqq \underline{\int_A} (\phi - \phi_n) \leqq \epsilon.$$

Hence from theorem 7,

$$\underline{\int_A} f - \overline{\int_A} f_n \leqq \epsilon,$$

and relation (1) is proved.

When the lower integral of ϕ is infinite λ may be chosen such that for M large at pleasure

$$\underline{\int_A} \phi_\lambda \geqq M.$$

Hence

$$\lim_{n=\infty} \overline{\int_A} f_n \geqq \lim_{n=\infty} \overline{\int_A} (f_n)_\lambda \geqq \underline{\int_A} f_\lambda \geqq \underline{\int_A} \phi_\lambda \geqq M$$

and

$$\lim_{n=\infty} \bar{\int}_A f_n = \infty.$$

This completes the proof of (1).

If we use the inequality $f_n - f \leqq \psi_n - \psi$, relation (2) is readily established by means of theorem 14.

THEOREM 17. *Let the convergence* of $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ to $f(x_1, \cdots, x_m)$ be monotone (for some points of A monotone and increasing and for some points monotone and decreasing). Then if the upper integral of $|f_1|$ exists,*

$$\lim_{n=\infty} \bar{\int}_A f_n \geqq \bar{\int}_A f \tag{1}$$

provided this limit and integral exist, finitely or infinitely. And further

$$\lim_{n=\infty} \underline{\int}_A f_n \leqq \bar{\int}_A f \tag{2}$$

provided the limit and integral exist, finitely or infinitely.

Define the auxiliary functions g and h as follows:

$g = f$ for $f > 0$, $g = 0$ for $f \leqq 0$; $h = -f$ for $f < 0$, $h = 0$ for $f \geqq 0$.

Then by theorem 16

$$\lim_{n=\infty} \bar{\int}_A g_n \geqq \underline{\int}_A g, \qquad \lim_{n=\infty} \underline{\int}_A h_n \leqq \bar{\int}_A h.$$

It follows from theorem 9 that when $\int_A f$ exists finitely or infinitely, either the lower integral of g or the upper integral of h is limited. Subtracting, we have by means of theorem 9

$$\operatorname{im}_{n=\infty} \bar{\int}_A f_n = \lim_{n=\infty} \bar{\int}_A (g_n - h_n) = \lim_{n=\infty} \left[\bar{\int}_A g_n - \underline{\int}_A h_n \right] \geqq \underline{\int}_A g - \bar{\int}_A h = \underline{\int}_A (g - h) = \underline{\int}_A f.$$

Relation (2) is established in a similar manner.

COROLLARY. *Let the convergence* of $f_1, f_2, \cdots$ to f be monotone. Then provided the integrals of f, f_2, f_3, $\cdots$ exist finitely or infinitely and the integral of f_1 exists finitely,*

$$\lim_{n=\infty} \int_A f_n = \int_A f.$$

THEOREM 18. *Let $f_1(x_1, \cdots, x_m)$, $f_2(x_1, \cdots, x_m)$, $\cdots$ be a sequence of functions integrable in A and converging to $f(x_1, \cdots, x_m)$. Let*

* This theorem is true for $m \geqq 2$. If $m = 1$, it is true if the integrals (finite or infinite) exist absolutely.

$$|f_n(x_1, \cdots, x_m)| < M, \qquad n \geqq n_0$$

where M is arbitrarily large but fixed. If

$$\lim_{n=\infty} \int_A f_n, \qquad \int_A f$$

exist, they are equal.

On denoting by A_n the partial aggregate of A for which $|f - f_m| \leqq \sigma$, where m takes on the values $n, n+1, \cdots$, it may be shown as in theorem 13 that for proper choice of n

$$\int_A |f - f_n| = \int_A |f - f_n| < \epsilon.$$

Hence

$$\left| \int_A f - \int_A f_n \right| = \left| \int_A (f - f_n) \right| < \epsilon,$$

and the theorem is proved.

A corresponding theorem for the differentiation of a sequence of functions may be easily derived.

THEOREM 19. *Let $f_1(x), f_2(x), \cdots$ be a sequence of differentiable functions in an interval (a, b) which converge to a differentiable limit. Let $f_1'(x), f_2'(x), \cdots$ be integrable and suppose also $\lim_{n=\infty} f_n'(x) \equiv \phi(x)$ to exist either finitely or infinitely in (a, b). If then either*

1) *$f_1', f_2', \cdots$ form a monotone sequence at each point of (a, b) (increasing at some or all points, decreasing at the points of the complementary aggregate, the infinite discontinuities in each case forming at most an aggregate of the first species);*

or 2) *$|f_i'(x)| \leqq G (i = 1, 2, \cdots)$, G being arbitrarily large but fixed; then for each point of continuity of ϕ,*

$$D_x(f) = \phi(x).$$

For by theorems 13 (corollary), 17, and 18, and the fundamental theorem of the integral calculus *

$$\int_a^x \phi(x) = \lim_{n=\infty} \int_a^x f_n'(x) = \lim_{n=\infty} \{ f_n(x) - f_n(a) \}.$$

Differentiating this expression, we have for each point of continuity of ϕ, †

$$\phi(x) = D_x [\lim_{n=\infty} f_n(x)] = D_x f.$$

* DE LA VALLÉE-POUSSIN, Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 430.

† *Lectures*, § 604.

§4. *Reduction of multiple integrals.*

The problem of the reduction of a multiple integral depends in large measure upon the nature of the field, and the arrangement of the discontinuities. If a function f is limited ($|f| < G$) throughout a discrete aggregate $\mathfrak{A}_1$, the multiple integral and the iterated integral of f over $\mathfrak{A}_1$ are both zero. For in this case the points of $\mathfrak{B}_1$ where $\bar{\mathfrak{C}}_1 \geqq \epsilon > 0$, or where

$$\int_{\mathfrak{C}_1}^{\bar{}} |f| > G\ \epsilon$$

form a discrete aggregate. This being true for every $\epsilon > 0$, we have

$$\left| \int_{\mathfrak{B}_1} \underline{\int}_{\mathfrak{C}_1}^{\bar{}} f \right| = \int_{\mathfrak{B}_1}^{\bar{}} \int_{\mathfrak{C}_1}^{\bar{}} |f| = 0 .$$

For an unlimited function this reasoning will no longer hold, as is seen by a reference to Example I below. The aggregate of points of $\mathfrak{B}_1$ where $\bar{\mathfrak{C}} > 0$ form an aggregate of measure zero (theorem 3), but this aggregate may be everywhere dense in $\mathfrak{B}_1$. Now if $\mathfrak{A}_1$ represents the discrete partial aggregate of $\mathfrak{A}$ where the integrable function has discontinuities equal to or greater than $\sigma > 0$, we see that if f is limited, the multiple integral may be replaced by the iterated integral. If, however, these discontinuities are infinite, this is not always possible. This is the fundamental difference between the problems of reduction of multiple integrals of limited and unlimited functions.

Before entering upon a discussion of the theory, it may be well to consider some examples.

Example I. Denote by R the rectangle $(1, 0), (1, 1)(-1, 1), (-1, 0)$, and by T those points whose coördinates are of the form $|x| \leqq 1/2^n$, $y = (2m+1)/2^n$ where m and n are positive integers. The aggregate T is discrete in two dimensions and hence the double integral of any function $f(x, y)$ over T is zero. Giving to $f(x, y)$ the value 2^n for the points of T, we note that

$$\int_{X_T} f(x, y) = 2, \qquad \int_{Y_T} \int_{X_T} f(x, y) = 2, \qquad \int_T f(x, y) = 0,$$

where X_T, Y_T denote the sections of the aggregate T. Assigning to $f(x, y)$ the value 2^n at the points T and the value zero at the remaining points of R, we have

$$\int_R f = 0, \qquad \underline{\int}_Y \int_X f = 0, \qquad \int_Y^{\bar{}} \int_X f = 2.$$

Example II. In the rectangle R of the preceding example assign to $f(x, y)$ the value 2^n for points of T, the value 1 for points whose ordinates are not of

the form $y=(2m+1)/2^n$ and the value 0 for the remaining points. The integral over each section X is 2, and hence the iterated integral

$$\int_Y \int_X f$$

exists. However, the integral

$$\int_Y \int_X f_\lambda$$

exists for no value of λ.

Example III. Assign to $f(x, y)$ the value $2^n(1/x \cos 1/x)$ at the points of T (Example I) and the value zero for the remaining points of R. Then since T is discrete in two dimensions,

$$\int_R f = \int_R |f| = 0.$$

We have also

$$\int_Y \int_X f = 0.$$

On the other hand

$$\int_X |f| = \infty$$

for all points of Y whose coördinates are of the form $y=(2m+1)/2^n$, and the iterated integral of $|f|$ does not exist.

Example IV. At the points of T (Example I) whose abscissæ are rational numbers let $f(x, y)=1$, and at the remaining points of T let $f(x, y)=0$. Then

$$\int_T f = \int_Y \underline{\int}_X f = 0.$$

Example V. At the points of T (Example I) at which $x>0$ let $f(x, y)=2^n$; where $x=0$ let $f(x, y)=0$; and where $x<0$ let $f(x, y)=-2^n$. Then

$$\int_R f = \int_Y \int_X f = 0.$$

Example VI. PRINGSHEIM* has defined an aggregate everywhere dense in two dimensions and yet having only a finite number of points on any line parallel to the X or Y axis. Denote by P the aggregate of such points contained in a unit square S. Assign to $f(x, y)$ the value $M>0$ throughout the aggregate P. Then

$$\int_P f = M, \qquad \int_{Y_P} \int_{X_P} f = \int_{X_P} \int_{Y_P} f = 0.$$

*Münchener Sitzungsberichte (1899), or PIERPONT's *Lectures*, p. 546.

If now we consider the square S and assign to f the value 0 at the points of S not belonging to P,

$$\int_X \int_Y f = \int_Y \int_X f = \underline{\int}_S f = 0, \qquad \bar{\int}_S f = M.$$

In his classic memoir * DE LA VALLÉE-POUSSIN has shown that in case the double integral and the iterated integral exist, a sufficient condition for their equality is the regular convergence of the integral

$$\int_X f(x, y)\, dx.$$

In undertaking to prove this condition necessary he restricts his discussion to the case of a function of one sign only. Let T be the fundamental rectangle with sides parallel to the axes. Denote by T_σ the points of T at which the discontinuity of f exceeds $\sigma > 0$, and by X_σ the points of T_σ in any section parallel to the x axis. Let Y_1 denote the partial aggregate of Y such that $\bar{X}_\sigma > 0$. If Y_1 is discrete, DE LA VALLÉE-POUSSIN proved that the condition of regular convergence is necessary. It may be shown that the points of Y where $\bar{X}_\sigma \geqq \delta > 0$ form a discrete aggregate and that consequently Y_1 is of measure zero. That Y_1 is not necessarily discrete is seen from example IV.

In 1899 DE LA VALLÉE-POUSSIN returned to the problem and devoted a memoir † to its discussion. He states that, although he has not been able to remove the restrictions cited, he believes them unnecessary. A minimum function mf is introduced which is defined at each point as the minimum value of f in the infinitesimal vicinity of that point. For a positive function it is shown that

$$(1) \qquad \bar{\int}_T f \geqq \underline{\int}_Y \bar{\int}_X mf \geqq \underline{\int}_X \underline{\int}_Y mf \geqq \underline{\int}_T f.$$

A comparison of (1) with the results of theorem 20 show that the introduction of the minimum function is unnecessary.

This problem was later discussed by SCHÖNFLIES,‡ his methods involving the introduction of a "most nearly continuous function," corresponding in some measure to the minimum function of DE LA VALLÉE POUSSIN. Although the double integral of the auxiliary function is equal to the double integral of f, the existence of the iterated integral of f is not a necessary condition for the existence and equality of the iterated integral of the "most nearly continuous function." The statement of SCHÖNFLIES that the convergence of

* Loc. cit.

† Journal de Mathématiques, ser. 5, vol. 5 (1899).

‡ *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* (1900), p. 210.

$$\int_Y \int_X f_\lambda$$

to the iterated integral of f is a simple process (though apparently a double limit) is open to criticism. Example I furnishes an illustration of the fact that the double integral and the iterated integral may exist in a field possessing content and that at the same time the values of these integrals may be different.

The results of DE LA VALLÉE-POUSSIN have been extended in a recent article by HOBSON.* He also introduces an auxiliary function. The field G which possesses content is enclosed in a rectangle, the auxiliary function f being equal to the original function ϕ at the points where the latter is defined, and equal to zero at the remaining points of the rectangle. There is no investigation of the relations between the integrals of f and ϕ over the rectangle and G respectively. It is evident that the double integrals are equal, but that this is not necessarily true for the iterated integrals is shown by Example I. For the auxiliary function f, HOBSON shows that in case it is positive the existence of the double integral and the iterated integrals is a sufficient condition for their equality.† To obtain this result there is no need of his generalization of the notion of regular convergence introduced by DE LA VALLÉE POUSSIN. A much more general result for a function of both signs is here obtained without it and is given in theorem 21.‡

When the double integral and the iterated integral exist by the definition of HARNACK, there is no necessity for any sort of convergence (regular, or regular except for a set of points of zero measure) of the integral

$$\int_X f_{\lambda_1 \lambda_2}.$$

This is readily seen from Example III.

We proceed now to build up the theory of reduction of multiple integrals for a limited field. It is assumed as already stated that A *possesses content* and that B either *possesses content* or *is closed.*§ No restriction is put on the sections $\mathfrak{C}$. *The results of this memoir are true if the integrals over* $\mathfrak{C}$ *exist non-*

*Proceedings of the London Mathematical Society, ser. 2, vol. 4 (1906), p. 139.

† This result was announced to the Society at the meeting in April, 1906. HOBSON'S article appeared in June, 1906.

‡ Since the definition of DE LA VALLÉE-POUSSIN is used, this integral can exist only absolutely. From this standpoint, it seems that the example given on page 157 of HOBSON'S article has no meaning. Since the integral

$$\int_0^1 \frac{1}{y} \sin \frac{1}{y}\, dy$$

exists only non-absolutely, it cannot illustrate any argument of his paper.

§ This restriction on A is necessary for a limited function f (Example VI), while that on B is necessary only when f is unlimited at some points of A.

absolutely,* in accordance with the definition ordinarily used. The theory, however, is developed here only when the integrals over $\mathfrak{C}$ are defined by the method of DE LA VALLÉE-POUSSIN.

THEOREM 20. *Let* $f(x_1, \cdots, x_m)$ *be a function defined over* A. *Then if* $f \geqq -G$,

$$\int_A^{\bar{}} f \geqq \underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f \geqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_A f. \tag{1}$$

If $f \leqq G$,

$$\int_A^{\bar{}} f \geqq \int_B^{\bar{}} \int_{\mathfrak{C}}^{\bar{}} f \geqq \int_B^{\bar{}} \underline{\int}_{\mathfrak{C}} f \geqq \underline{\int}_A f. \tag{2}$$

To deduce (1), let $\lambda_1 < \lambda_2 < \cdots$ be a series of values increasing to infinity. Then the functions

$$\int_{\mathfrak{C}}^{\bar{}} f_{\lambda_1}, \quad \int_{\mathfrak{C}}^{\bar{}} f_{\lambda_2}, \quad \cdots; \qquad \underline{\int}_{\mathfrak{C}} f_{\lambda_1}, \quad \underline{\int}_{\mathfrak{C}} f_{\lambda_2}, \quad \cdots$$

form monotone increasing sequences which by definition converge to

$$\int_{\mathfrak{C}}^{\bar{}} f, \quad \underline{\int}_{\mathfrak{C}} f.$$

For any value of λ it follows from theorem 12 that

$$\int_A^{\bar{}} f_\lambda \geqq \int_B^{\bar{}} \int_{\mathfrak{C}}^{\bar{}} f_\lambda; \qquad \underline{\int}_B \underline{\int}_{\mathfrak{C}} f_\lambda \geqq \underline{\int}_A f_\lambda.$$

A passage to the limit gives by means of the extension of theorem 13 for $f \geqq -G$ the following relations:

$$\int_A^{\bar{}} f \geqq \lim_{\lambda=\infty} \int_B^{\bar{}} \int_{\mathfrak{C}}^{\bar{}} f_\lambda \geqq \underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f, \tag{3}$$

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} f \geqq \lim_{\lambda=\infty} \underline{\int}_B \underline{\int}_{\mathfrak{C}} f_\lambda \geqq \underline{\int}_A f. \tag{4}$$

Since, however,

$$\underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f \geqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} f,$$

relations (3) and (4) give (1). In a similar manner (2) is established. It may be remarked that relations (1) and (2) are true if one or more of the integrals are infinite.

COROLLARY 1. *If* $f \geqq -G$ *is integrable in* A,

$$\int_A f = \underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f.$$

* P. 345.

COROLLARY 2. *If $f \geqq -G$ is integrable in A and the iterated integral also exists, then*

$$\int_A f = \int_B \underline{\int}_{\mathfrak{C}} f.$$

COROLLARY 3. *If the integral of $f \geqq -G$ in A is infinite, then both the integrals*

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} f, \qquad \overline{\int}_B \underline{\int}_{\mathfrak{C}} f$$

are infinite.

COROLLARY 4. *If for $f \geqq -G$ one of the integrals*

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} f, \quad \underline{\int}_B \overline{\int}_{\mathfrak{C}} f, \quad \overline{\int}_B \underline{\int}_{\mathfrak{C}} f, \quad \overline{\int}_B \overline{\int}_{\mathfrak{C}} f \tag{5}$$

is limited, then

$$\underline{\int}_A f$$

is limited. If one of the integrals (5) *is unlimited, then the integral*

$$\overline{\int}_A f$$

is unlimited.

COROLLARY 5. *If for any $\epsilon > 0$ the points where the discontinuity of $f \geqq -G$ exceeds ϵ form a discrete aggregate, and if one of the integrals* (5) *is limited, then the multiple integral exists and*

$$\int_A f = \underline{\int}_B \underline{\int}_{\mathfrak{C}} f. \tag{6}$$

For since the discontinuities of f form a discrete aggregate,

$$\overline{\int}_A f_\lambda = \underline{\int}_A f_\lambda = \int_A f_\lambda.$$

By letting λ increase indefinitely it follows from the preceding corollary that the integral of f in A exists. Relation (6) then results by corollary 1.

THEOREM 21. *Let $f(x_1, \cdots, x_m)$ take on both positive and negative values in A. If*

$$\int_A f, \quad \int_B \underline{\int}_{\mathfrak{C}} f$$

exist, they are equal.

Since f is integrable in A, it is absolutely integrable, and by the corollaries of theorems 20 and 12 we have for sufficiently great values of λ, λ_1, λ_2

$$\epsilon > \int_A |f| - \int_A |f|_\lambda = \underline{\int}_B \underline{\int}_{\mathfrak{C}} |f| - \int_B \underline{\int}_{\mathfrak{C}} |f|_\lambda, \tag{1}$$

$$\epsilon > \left| \int_A f - \int_A f_{\lambda_1\lambda_2} \right| = \left| \int_A f - \int_B \underline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right|. \tag{2}$$

The functions f, $f_{\lambda_1\lambda_2}$ for any given point of A have the same sign and hence

$$|f - f_{\lambda_1\lambda_2}| = |f| - |f_{\lambda_1\lambda_2}|. \tag{3}$$

Now by means of theorems 8 and 10

$$\left| \int_B \underline{\int}_{\mathfrak{C}} f - \int_B \underline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| = \left| \int_B \left[\underline{\int}_{\mathfrak{C}} f - \underline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right] \right| = \left| \int_B \underline{\int}_{\mathfrak{C}} (f - f_{\lambda_1\lambda_2}) \right| \leqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} |f - f_{\lambda_1\lambda_2}|.$$

Applying equation (3) and theorem 8, we obtain

$$\left| \int_B \underline{\int}_{\mathfrak{C}} f - \int_B \underline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| \leqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} (|f| - |f|_\lambda) = \underline{\int}_B \left[\underline{\int}_{\mathfrak{C}} |f| - \underline{\int}_{\mathfrak{C}} |f|_\lambda \right], \tag{4}$$

where λ is the smaller of the two numbers λ_1, λ_2. Since the integral

$$\int_B \underline{\int}_{\mathfrak{C}} |f|_\lambda$$

exists, the last relation in corollary 2 of theorem 7 gives the equation

$$\underline{\int}_B \left[\underline{\int}_{\mathfrak{C}} |f| - \underline{\int}_{\mathfrak{C}} |f|_\lambda \right] = \underline{\int}_B \underline{\int}_{\mathfrak{C}} |f| - \int_B \underline{\int}_{\mathfrak{C}} |f|_\lambda.$$

Hence from (1) and (4)

$$\left| \int_B \underline{\int}_{\mathfrak{C}} f - \int_B \underline{\int}_{\mathfrak{C}} f_{\lambda_1\lambda_2} \right| \leqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} |f| - \int_B \underline{\int}_{\mathfrak{C}} |f|_\lambda < \epsilon. \tag{5}$$

Combining (2) and (5), we obtain

$$\left| \int_A f - \int_B \underline{\int}_{\mathfrak{C}} f \right| < 2\epsilon$$

and the theorem is proved.

COROLLARY. *Let the points where the discontinuity of f exceeds $\epsilon > 0$ form a discrete aggregate. Let*

$$\underline{\int}_B \underline{\int}_{\mathfrak{C}} |f|$$

exist. Then f is integrable in A.

Since the points of discontinuity of $f_{\lambda_1\lambda_2}$ form a discrete aggregate,

$$\bar{\int}_A f_{\lambda_1\lambda_2} = \underline{\int}_A f_{\lambda_1\lambda_2} = \int_A f_{\lambda_1\lambda_2}.$$

Now

$$\left| \int_A f_{\lambda_1\lambda_2} \right| = \left| \int_A g_{\lambda_1} - \int_A h_{\lambda_2} \right| \leqq \int_A (g_{\lambda_1} + h_{\lambda_2}) = \int_A |f_{\lambda_1\lambda_2}| \leqq \underline{\int}_A |f|.$$

The lower integral of $|f|$ is limited (corollary 4, theorem 20) and on passing to the limit for $\lambda_1 = \infty$, $\lambda_2 = \infty$, the corollary is established.

Upon assuming that the upper integral of f exists finitely or infinitely, various relations may be derived. By means of theorems 9, 12 and 7 we have the relation

$$\bar{\int}_A f_{\lambda_1\lambda_2} = \bar{\int}_A g_{\lambda_1} - \underline{\int}_A h_{\lambda_2} \geqq \bar{\int}_B \bar{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_B \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \geqq \bar{\int}_B \left[\bar{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \right].$$

On passing to the limit for $\lambda_1 = \infty$ and placing $f_{\lambda_2} = \lim_{\lambda_1=\infty} f_{\lambda_1\lambda_2}$, it follows from theorem 13 that

$$(6) \quad \bar{\int}_A f_{\lambda_2} \geqq \lim_{\lambda_1=\infty} \bar{\int}_B \left[\bar{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \right] \geqq \underline{\int}_B \lim_{\lambda_1=\infty} \left[\bar{\int}_{\mathfrak{C}} g_{\lambda_1} - \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \right] = \underline{\int}_B \left[\bar{\int}_{\mathfrak{C}} g - \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \right].$$

And by theorem 9

$$(7) \quad \underline{\int}_B \left[\bar{\int}_{\mathfrak{C}} g - \underline{\int}_{\mathfrak{C}} h_{\lambda_2} \right] = \underline{\int}_B \bar{\int}_{\mathfrak{C}} (g - h_{\lambda_2}) = \underline{\int}_B \bar{\int}_{\mathfrak{C}} f_{\lambda_2}.$$

Hence from (6) and (7)

$$\bar{\int}_A f_{\lambda_2} \geqq \underline{\int}_B \bar{\int}_{\mathfrak{C}} f_{\lambda_2}.$$

Finally by passing to the limit for $\lambda_2 = \infty$ we obtain

$$\int_A^{\bar{}} f \geqq \lim_{\lambda_2=\infty} \underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f_{\lambda_2} \geqq \underline{\int}_B \lim_{\lambda_2=\infty} \int_{\mathfrak{C}}^{\bar{}} f_{\lambda_2}. \tag{8}$$

If the integral

$$\underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f$$

exists finitely or infinitely, relation (8) gives

$$\int_A^{\bar{}} f \geqq \underline{\int}_B \lim_{\lambda_2=\infty} \int_{\mathfrak{C}}^{\bar{}} f_{\lambda_2} = \underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f.$$

If $\mathfrak{C}$ is a one-dimensional aggregate, it is evident that the upper integral of f over $\mathfrak{C}$ may exist non-absolutely for an everywhere dense set of points of B. For these sections the expressions

$$\lim_{\lambda_2=\infty} \int_{\mathfrak{C}}^{\bar{}} f_{\lambda_2}, \qquad \int_{\mathfrak{C}}^{\bar{}} f$$

have different values. If we denote by

$$\int_{\mathfrak{C}}^{\bar{\bar{}}} f$$

the maximum value obtained from

$$\int_{\mathfrak{C}}^{\bar{}} f_{\lambda_1 \lambda_2}$$

by letting λ_1, λ_2 increase indefinitely in any manner whatsoever, we have

$$\int_A^{\bar{}} f \geqq \underline{\int}_B \int_{\mathfrak{C}}^{\bar{\bar{}}} f.$$

We have then the exceedingly general result:

THEOREM 22. *Let the upper integral of f exist finitely or infinitely in A Then*

$$\int_A^{\bar{}} f \geqq \underline{\int}_B \int_{\mathfrak{C}}^{\bar{\bar{}}} f.$$

And further,

$$\int_A^{\bar{}} f \geqq \underline{\int}_B \int_{\mathfrak{C}}^{\bar{}} f \geqq \underline{\int}_B \underline{\int}_{\mathfrak{C}} f,$$

provided the last two integrals exist finitely or infinitely.

Corresponding relations exist when it is assumed that the lower integral of f exists finitely or infinitely. It may be noted that theorem 21 follows as a corollary to this theorem.

§ 5. *Integration and differentiation of series.*

Let $u_1(x_1, \cdots, x_m)$, $u_2(u_1, \cdots, x_m) \cdots$ be a sequence of functions defined over a limited field A which either possesses content or is closed. Unless otherwise stated no restriction is put on the discontinuities of the functions $u_1, u_2, \cdots$, which may take on either finite or infinite values. Form the sum

$$U_n(x_1, \cdots, x_m) = u_1(x_1, \cdots, x_m) + u_2(x_1, \cdots, x_m) + \cdots + u_n(x_1, \cdots, x_m).$$

For a point P of A chosen at pleasure and then fixed we shall assume that when n increases indefinitely, $U_n(P)$ converges toward a limit, finite or definitely infinite. Denote the aggregate of limiting values by U and write

$$U(x_1, \cdots, x_m) = \lim_{n=\infty} U_n(x_1, \cdots, x_m) = u_1 + u_2 + \cdots. \tag{1}$$

The object of the following discussion is the determination of some simple conditions under which the series (1) can be integrated term by term; in other words, under which

$$\lim_{n=\infty} \int_A U_n = \int_A \lim_{n=\infty} U_n = \int_A U.$$

The conditions under which this inversion of the order is permissible have been investigated by Osgood, Arzelà and others. Uniform convergence in a more or less modified form is shown to be a sufficient condition. In this section it is shown that for certain classes of functions the notion of uniformity is superfluous.

Theorem 23. *Let all terms of the series $U(x_1, \cdots, x_m)$ be of like sign and integrable in A. If $u_i(x_1, \cdots, x_m) \geqq 0$*

$$\underline{\int}_A U = \int_A u_1 + \int_A u_2 + \cdots,$$

and if $u_i(x_1, \cdots, x_m) \leqq 0$

$$\bar{\int}_A U = \int_A u_1 + \int_A u_2 + \cdots.$$

Since the sequence of functions $U_1, U_2, \cdots$ is monotone, this theorem is the same as the corollary of theorem 13. The integral on the left may be finite or infinite.

THEOREM 24. *Let*

$$\int_A |u_1| + \int_A |u_2| + \cdots$$

be finite. Then

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots,$$

provided these integrals exist.

Denoting by V the sum

$$\sum_{i=1}^{\infty} |u_i(x_1, \cdots, x_m)|,$$

we have from the preceding theorem and the second corollary of theorem 7

$$\lim_{n=\infty} \underline{\int}_A \left[V - \sum_{i=1}^{n} |u_i| \right] = \lim_{n=\infty} \left[\underline{\int}_A V - \sum_{i=1}^{n} \int |u_i| \right] = 0.$$

Now

$$\left| \int_A U - \sum_{i=1}^{n} \int_A u_i \right| = \left| \int_A \left[U - \sum_{i=1}^{n} (u_i) \right] \right| \leqq \underline{\int}_A \left[V - \sum_{i=1}^{n} |u_i| \right].$$

By passing to the limit for $n = \infty$, we have the theorem:

THEOREM 25. *Let $U(x_1, \cdots, x_m)$ be a series for which there exists a fixed number G and a positive integer ν such that $|U_n| < G$ for $n > \nu$. If $U, u_1, u_2, \cdots$ are integrable in A,*

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots.$$

This is the same as theorem 18.

THEOREM 26. *At each point of A let all the functions $u_1, u_2, \cdots$ have the same sign (at some points of A positive and at others negative). Then provided the integrals (finite or infinite) of $U, u_1, u_2, \cdots$ exist absolutely,*

$$\int_A U = \int_A u_1 + \int_A u_2 + \cdots.$$

This follows immediately from the corollary to theorem 17.

THEOREM 27. *Let $U(x) = u_1(x) + u_2(x) + \cdots$, be a series of functions defined in the interval (a, b). Let each of the terms $u_i(x)$ be differentiable and denote by $V(x)$ the series of derived terms $u_1'(x) + u_2'(x) + \cdots$. Let $V(x)$ and $u_i'(x)$ be integrable in (a, b). Let either* 1) *all the functions $u_i'(x)$ have the same sign at each particular point of A (at all points positive or at all points negative or at some points positive and at some negative), the*

points of infinite discontinuity of each term forming at most an aggregate of the first species; or 2)

$$\sum_{i=1}^{\infty} \int_a^b |u_i'(x)|$$

be limited; or 3) $|\sum_{i=1}^{n} u_i'(x)| < G$ *for all values of* n, G *being arbitrarily large but fixed. Then for each point of continuity of* V

$$D_x U(x) = V(x).$$

By theorems 23, 24, 25, 26 and the fundamental theorem of the integral calculus *

$$\int_a^x V(x) = \sum_{i=1}^{\infty} \int_a^x u_i'(x)\,dx = \sum_{i=1}^{\infty} [u_i(x) - u_i(a)] = U(x) - U(a).$$

Differentiating this expression we have for every point of continuity † of V

$$V(x) = D_x U(x).$$

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November, 1907.

CORRECTION. The author wishes to call attention to an error which occurs in a former paper [Transactions, vol. 7 (1906), p. 451]. In the proof there of theorem 3 (3°), it is erroneously assumed that the limit of the upper content $\bar{B}_i$ of a sequence of aggregates $B_1 \geqq B_2 \geqq \cdots$ is zero provided that these aggregates have no point in common. This error causes an hiatus in the proof of theorem 6 of that paper. To correct this proof let the part after relation (2) on page 452 be neglected and replaced by the following discussion.

Denote by $\mathfrak{C}$ the points of $\mathfrak{B}$ where $f \geqq 0$. Let $\Delta_1, \Delta_2, \cdots$ be a set of superimposed divisions whose norms converge to zero. Denote by $\mathfrak{B}_k$ all points of $\mathfrak{B}$ in the cells of Δ_k which contain at least one point of $\mathfrak{C}$. The norm of the division may be so taken that the difference of the upper contents of $\mathfrak{C}$ and

* DE LA VALLÉE-POUSSIN, Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 430.
† *Lectures*, § 604.

$\mathfrak{B}_k$ is small at pleasure. Since f is limited throughout $\mathfrak{B}_k$ it follows as a corollary to theorem 4 that for proper choice of k

$$\left|\bar{\int}_{\mathfrak{B}_k} f - \bar{\int}_{\mathfrak{C}} f\right| < \frac{\epsilon}{2}.$$

Combined with (2) of page 452 this gives

$$\bar{\int}_{\mathfrak{C}} f < \epsilon.$$

Hence by theorem 5,

$$\bar{\int}_{\mathfrak{B}} g < \epsilon.$$

Likewise it may be shown that

$$\bar{\int}_{\mathfrak{B}} h < \epsilon.$$

Then

$$\underline{\int}_{\mathfrak{B}} |f| \leqq \bar{\int}_{\mathfrak{B}} |f| = \bar{\int}_{\mathfrak{B}} (g + h) \leqq \bar{\int}_{\mathfrak{B}} g + \int_{\mathfrak{B}} h < 2\epsilon.$$